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Convergence Theorem for Non-commutative Feynman Graphs and Renormalization

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Abstract

We present a rigorous proof of the convergence theorem for the Feynman graphs in arbitrary massive Euclidean quantum field theories on non-commutative $\mathbb{R}^d$ (NQFT). We give a detailed classification of divergent graphs in some massive NQFT and demonstrate the renormalizability of some of these theories.

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1 Introduction

Recently, perturbative aspects of non-commutative field theories (NQFT) have received much attention [2-12].[1] A list of references directly relevant for this work together with a short summary of the corresponding relevant results is

[1] For the non-perturbative aspects of NQFT see ref.[13].

• Ref.[4]. Formulates a general convergence theorem for the non-commutative Feynman graphs.

• Ref.[5]. Finds explicit examples of divergent non-planar graphs in the massive scalar NQFT. Depending on the order of integration in the Feynman multiple integrals, the divergences show up either as infrared (IR) or ultraviolet (UV). Ref.[5] interprets these divergences as IR.[2]

• Ref.[6]. Shows that the massive ϕ^4 NQFT is renormalizable up to two loops. Ref.[7]. Shows that the massive complex scalar NQFT with the interaction term $\phi^* \star \phi \star \phi^* \star \phi$ is one-loop renormalizable and the one-loop two-point function is free of IR singularities.

In ref.[4], based on various consistency arguments we formulated a convergence theorem for the Feynman integrals in massive scalar NQFT with non-derivative couplings. In this paper we give a rigorous proof of the theorem for arbitrary massive NQFT on $\mathbb{R}^d$.

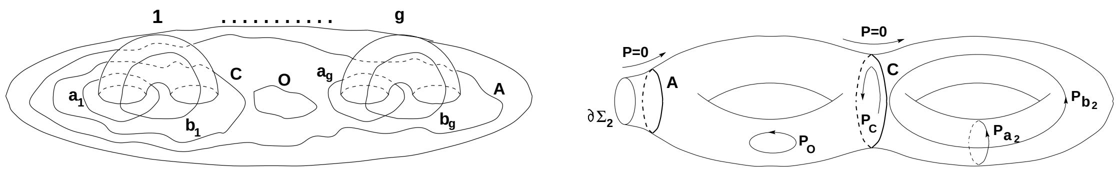
The paper is organized as follows. In section 2 we give a refined version of the heuristic proof of the convergence theorem given in ref.[4]. The precise statement of the convergence theorem and the relevant definitions are given in section 3. In section 4 we analyze the structure of divergences in some massive NQFT and give diagrammatic arguments for the renormalizability of some of these theories. In section 5 we cast the proof of the convergence theorem for the commutative scalar theories with non-derivative couplings in a form that admits generalization to the non-commutative case. In section 6 we prove the convergence theorem for the scalar NQFT with non-derivative couplings. Using some of the constructions, theorems and lemmas from section 6, in section 7 we prove the convergence theorem for arbitrary NQFT on $\mathbb{R}^d$ with massive propagators. Section 8 contains some comments. The sections are inter-related as follows.

$$\begin{array}{c} \text{section 4} \\ \uparrow \\ \text{sections 2 and 3} \\ \downarrow \\ \text{section 5} \rightarrow \text{section 6} \rightarrow \text{section 7} \end{array}$$

2 A heuristic proof of the convergence theorem for the non-derivative scalar theories

In this section we give a refined version of the heuristic proof of the convergence theorem given in ref.[4].

[2]As we will see in section 4, there are certain types of divergent graphs with the property: (1) in the corresponding Feynman multiple integrals the divergences always show up as UV, independent of the order of the integration and (2) the divergences come from the non-planar subgraphs.



(a)A genus g surface with a boundary **(b)**Momentum flow on a $g = 2$ surface Σ_2

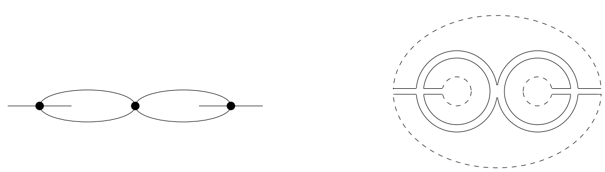
Figure 1: Two-surface with a boundary.

Consider a 1PI Feynman graph G in a non-derivative massive scalar quantum field theory on non-commutative $\mathbb{R}^d$. It can be drawn on a 2-surface $\Sigma_g(G)$ of genus g with a boundary $\partial\Sigma_g$. If G has no external lines, then $\Sigma_g(G)$ does not have a boundary. In general $\partial\Sigma_g$ consists of several components. $\Sigma_g(G)$ for the graph in figure 2(a) has three holes (boundary components). This can be seen from the corresponding ribbon graph in figure 2(b). The Euler characteristic χ is given by

$$\chi = 2 - 2g - B = C - I + V \tag{2.1}$$

where g is the number of handles, B is the number of holes, I is the number of edges (including external lines), V is the number of vertices and C is the number of closed single lines. For the graph in figure 2(b) we have $g = 0$, $B = 3$, $I = 8$, $V = 3$ and $C = 4$.

The non-trivial cycles of Σ_g are $a_1, b_1, \ldots, a_g, b_g$ (see figure 1(a)). Cycles A, C and 0 are trivial.[3]



(a) (b)

Figure 2: Sphere with three holes.

The convergence theorem of ref.[4] can be stated as follows.

[3]A cycle on Σ_g is called non-trivial if it is a non-trivial element of the first homology group $H_1(\Sigma_g)$. In addition to the trivial cycles that are contractible to a point, there are trivial cycles which are not contractible to a point. For example, cycles A and C in figure 1 are trivial because $A = a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1}$ and $C = a_1 b_1 a_1^{-1} b_1^{-1}$, i.e. A and C are commutants. The fundamental group $\mathcal{F}(\Sigma_g)$ is a free group generated by the generators a_i, b_i $(i = 1, \ldots, g)$ satisfying the relation $\prod_{i=1}^{g} a_i b_i a_i^{-1} b_i^{-1} = 1$. The commutator group $[\mathcal{F}, \mathcal{F}]$ is generated by the products of commutants $ABA^{-1}B^{-1}$, $A, B \in \mathcal{F}$. The first homology group $H_1(\Sigma_g)$ is the quotient $H_1(\Sigma_g) = \mathcal{F}/[\mathcal{F}, \mathcal{F}]$. See ref.[14] for the details.

A 1PI graph G is convergent for non-exceptional external momenta if for any subgraph $\gamma \subseteq G$ (possibly disconnected) at least one of the following conditions is satisfied:

(1) $\omega(\gamma) - c_G(\gamma)d < 0$,

(2) $j(\gamma) = 1$.

where $c_G(\gamma)$ is the number of non-trivial homology cycles of $\Sigma(G)$ wrapped by the subgraph γ, $j(\gamma)$ is an index which characterizes the non-planarity of γ with respect to the external lines of graph G, and $\omega(\gamma)$ is the superficial degree of divergence of a graph γ:

$$\omega(\gamma) = dL(\gamma) - 2I(\gamma), \tag{2.2}$$

where L and I are the number of independent loops and internal lines of γ respectively. The graph in figure 2 has $j(G) = 1$. The definition of the index j will be given below.

Let $I_{mn}(G)$ be the intersection matrix of the internal lines of G.[4] The phase factor associated with the internal lines of the Feynman graph G is[1, 2]

$$\exp\ (i\phi(p)) = \exp\ \left(i \sum_{m,n} I_{mn}(G) \Theta_{\mu\nu} p_m^\mu p_n^\nu \right) \tag{2.3}$$

where p_m are the internal momenta. The matrix $I_{mn}(G)$ is not a topologically invariant quantity. Namely, there may be several different intersection matrices for a given graph G. The phase factor Eq. (2.3) gives rise to a unique topology of G. Thus Eq. (2.3) should be rewritten in more invariant terms. For this purpose let us analyze the momentum flow on the surface $\Sigma(G)$. Figure 1(b) illustrates the flow of momentum on a genus two surface with a boundary. There are *topologically trivial* flows such as p_0 and *topologically non-trivial* flows such as p_A, p_C, p_{a_2} and p_{b_2}. Note that the momentum flowing along the cycles is defined as in figures 3 and 4. Since the total external momentum flowing into the surface Σ_2 through $\partial\Sigma_2$ is zero, the net momentum flowing across A and C is zero (the momenta p_A, p_C along A and C are nonzero). The phase factor associated with a graph arises from the *linking* of topologically non-trivial flows. In figure 1(b), p_{a_2} and p_{b_2} contribute a phase factor $\exp(i\theta_{\mu\nu} p_{a_2}^\mu p_{b_2}^\nu)$. Cycles A and C do not contribute to the phase factor because the net momentum flowing across each of these cycles is zero. Since the cycles a_2, b_2 are homologically non-trivial and the cycles A, C and 0 are homologically trivial, we conclude that *only the momentum flow along the homologically non-trivial cycles contribute to the phase factor.*

The phase factors are supposed to regulate the subgraphs of G and the above argument suggests that only those subgraphs which wrap homologically non-trivial cycles are

[4]It is constructed as follows. On $\Sigma_g(G)$ choose an arbitrary tree T of G. Now shrink T to a point. The resulting graph G/T is a genus g single-vertex graph. Next remove all the loops of G/T which wrap homologically trivial cycles of $\Sigma(G)$. The $I_{mn}(G)$ can be read-off from the resulting graph. Note that $I_{mn}(G)$ depends on the choice of surface $\Sigma(G)$ and tree T. Thus $I_{mn}(G)$ is not an invariant quantity. The procedure for constructing $I_{mn}(G)$ described here in Riemann surface-theoretic terms is equivalent to the one described in refs.[1, 2] in a different way.

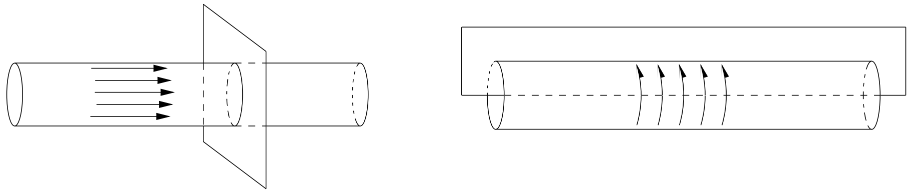


Figure 3: Measurement of momentum flow.

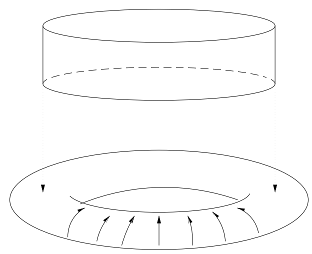


Figure 4: Measurement of momentum flow along a-cycle.

regulated by the phase factors. This explains why the cycle number $c_G(\gamma)$ in the condition 1 of the convergence theorem is defined to be the number of homologically non-trivial cycles wrapped by γ.

The condition $\omega(G) - c(G)d < 0$ for G can most easily be understood from the Schwinger representation of the Feynman integral I_G for G at zero external momenta. Schematically, this representation reads

$$I_G = \int d\alpha_1 \cdots d\alpha_I \frac{\exp(-\sum_l \alpha_l m_l^2)}{(P(G,\theta))^{\frac{d}{2}}} \tag{2.4}$$

where I is the number of internal lines of G and

$$P(G,\theta) = \sum_{n=0}^{\frac{c(G)}{2}} \theta^{2n} P_{2n}(\alpha), \qquad P_{2n} = \sum_{\{i_1,i_2,\ldots,i_{L-2n}\}} \alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_{L-2n}} \tag{2.5}$$

is a polynomial of degree $c(G)$ in θ. $c(G)$ is simply twice the genus of G. Rescaling the Schwinger parameters, $\alpha_i \to t\alpha_i$, we find for $t \sim 0$

$$I_G \sim t^{I-\frac{d}{2}(L-c(G))} \tag{2.6}$$

Thus we see that I_G is overall convergent if $\omega(G) - c(G)d < 0$.

The condition $j(\gamma) = 1$ can be understood as follows. Consider the subgraph γ of the $g = 1$ graph with five holes shown in figure 5. Let p_γ be the momentum flowing along γ. Let p_1, p_2 and p_3 be the total external momentum flowing into the external lines attached

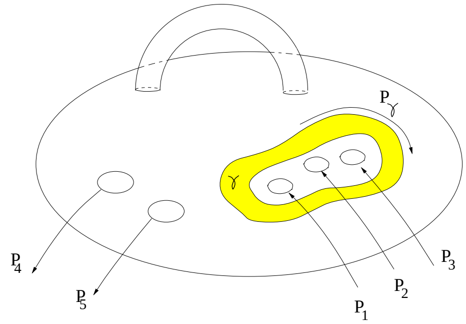


Figure 5: Index j.

to the holes. The phase factor associated with the subgraph γ is

$$\exp\left(i(p_1 + p_2 + p_3) \wedge p_\gamma\right) \tag{2.7}$$

where $p \wedge q \equiv p_\mu \Theta_{\mu\nu} q_\nu$. This phase factor will regulate γ for the arbitrary $\omega(\gamma)$ as long as $p_1 + p_2 + p_3 \neq 0$. If a subgraph has at least one hole inside and one hole outside of it as γ from figure 5 does, then $j(\gamma) = 1$. Otherwise, $j(\gamma) = 0$. A mathematically more precise definition of j will be given in section 3. Note that when

$$p_1 + p_2 + p_3 = 0 \tag{2.8}$$

the phase factor in Eq. (2.7) does not regulate γ. The momenta satisfying the relation Eq. (2.8) are called exceptional.

In the Schwinger representation of the Feynman integral the condition $j(G) = 1$ shows up as follows. For the non-exceptional external momenta, there will be a factor

$$\exp(-Q(p, \theta))$$

in the integrand in Eq. (2.4) which will scale like

$$\exp(-\frac{f(p)}{t}), \quad f(p) > 0$$

if $j(G) = 1$ and make the integral overall convergent at $t = 0$.

Remark (on the commutants). The graph shown in figure 6 contains the subgraph γ formed by the lines 1 and 2 which wraps the cycle $a_1 b_1 a_1^{-1} b_1^{-1}$. The latter cycle is a commutant. From the figure it is clear that $p + q = 0$. The subgraph γ considered as a graph on its own is a sphere with two holes (see figure 7). $p + q = 0$, which is the case of the exceptional momenta, means that the total momentum flowing into the hole is zero. A more sophisticated example of a commutant is given in figure 8. The graph in figure 9(a) contains a subgraph γ formed by the lines 1,2,8 and 9 which wraps the cycle $b_1 a_2 b_1^{-1} a_2^{-1}$ (see figure 9(b)). As it is clear from figure 9(b), γ is a sphere with two holes and the total momentum flowing into the hole is zero.

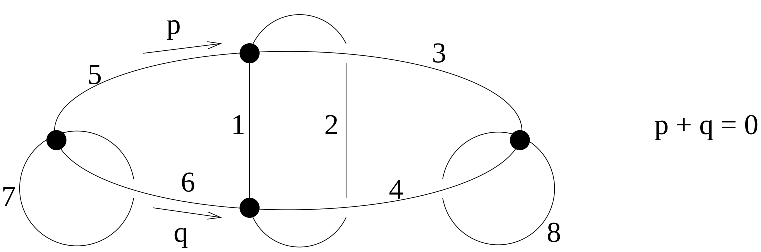


Figure 6: An example of the commutant $a_1 b_1 a_1^{-1} b_1^{-1}$.

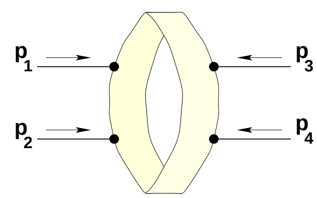


Figure 7: A ring with four external lines.

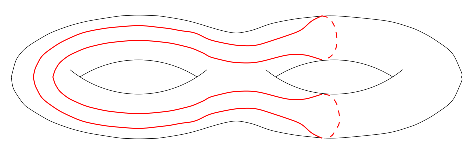


Figure 8: Commutant $b_1 a_2 b_1^{-1} a_2^{-1}$.

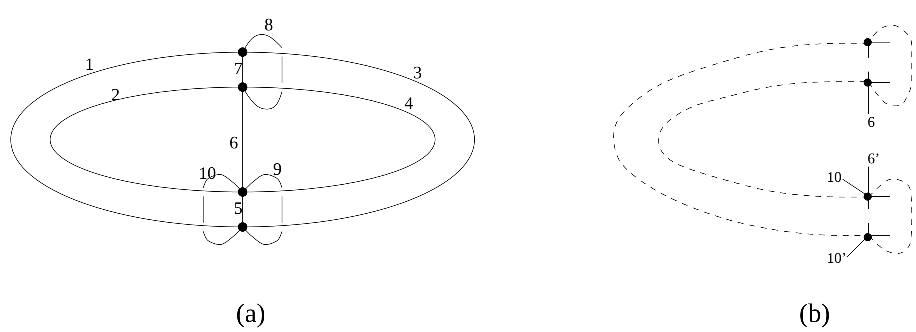


Figure 9: An example of the commutant $b_1 a_2 b_1^{-1} a_2^{-1}$.

3 Statement of theorem and definitions

Let us give some definitions required for the formulation of the convergence theorem. The examples illustrating some of these definitions can be found in ref.[4]. In what follows, unless otherwise stated, by a graph we mean an arbitrary (possibly disconnected) graph.

Let G be a connected NQFT Feynman graph. It can be drawn on a 2-surface $\Sigma_g(G)$ of genus g with a boundary $\partial\Sigma_g$. Let us define the cycle number $c_G(\gamma)$ of an arbitrary subgraph $\gamma \subseteq G$ with respect to the surface $\Sigma(G)$.

Definition 1. *The first homology group of $\Sigma_g(G)$ for the connected graph G has the basis $\{a_1, b_1, \ldots, a_g, b_g\}$ (see figure 1). $c_G(\gamma)$ is defined as the number of inequivalent non-trivial cycles of $\Sigma_g(G)$ spanned by the closed paths in γ.*

Remark. One should distinguish between *abstract* cycles which are not wrapped by any loop of the graph and *physical* cycles which are wrapped by the loops of the graph. To make this point clear consider a genus g graph G. There may be a number of ways of drawing G and thus there may be more than one genus g surface associated with G. Let us consider one particular surface Σ_g. Let $\{a_1, b_1, \ldots, a_g, b_g\}$ be the canonical basis of the homology group associated with Σ_g. It is clear that a_i, b_i are not necessarily wrapped by any loop of G. For example, it may well happen that $a_1 b_1$ and $a_1 b_1^{-1}$ are wrapped by loops in G, but a_1 and b_1 are not. Given a loop $\mathcal{L}$ in a graph G, we will denote by $\mathcal{C}(\mathcal{L})$ the corresponding homology cycle of $\Sigma(G)$. As an example consider the graph in figure 9(a). Let $\{1, 3\}$ and $\{2, 4\}$ be the subgraphs formed by the lines 1,3 and 2,4, respectively. Then $\mathcal{C}(\{1, 3\}) = \mathcal{C}(\{2, 4\}) = b_1 b_2$. Note that $\mathcal{C}(\mathcal{L})$ is defined to be an *element* of the homology group. For example consider the graph in figure 6. It is clear that $\mathcal{C}(\{1, 3, 4\}) = \mathcal{C}(\{2, 3, 4\})$, since $\{1, 3, 4\}$ and $\{2, 3, 4\}$ differ only by a commutant, i.e. a trivial element of the homology group.

We will occasionally drop the index G in c_G when the surface $\Sigma(G)$ in question is clear from the context.

Remark. A general NQFT Feynman graph G with N_c connected components can be drawn on N_c 2-surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_{N_c}}$. The cycle number $c_G(\gamma)$ of an arbitrary subgraph $\gamma \subseteq G$ is defined as a sum of cycle numbers of γ with respect to the surfaces $\Sigma_{g_1}, \ldots, \Sigma_{g_{N_c}}$. In what follows, by the genus g of a disconnected graph we mean the sum of genera of the connected components.

Let $\mathcal{E}(G)$ be the set of external lines of the graph G. Consider two external lines $m, n \in \mathcal{E}(G)$. As in figure 10 set the rest of the external momenta of G to zero and connect the lines m and n. Denote the resulting graph by G_{mn}. Let $c_{G_{mn}}(\gamma)$ be the cycle number of an arbitrary subgraph $\gamma \subset G$ with respect to the 2-surface of graph G_{mn}. There are only two possibilities:

$$c_{G_{mn}}(\gamma) > c_G(\gamma) \qquad \text{or} \qquad c_{G_{mn}}(\gamma) = c_G(\gamma)$$

The index j of an arbitrary subgraph $\gamma \subseteq G$ is defined as follows.

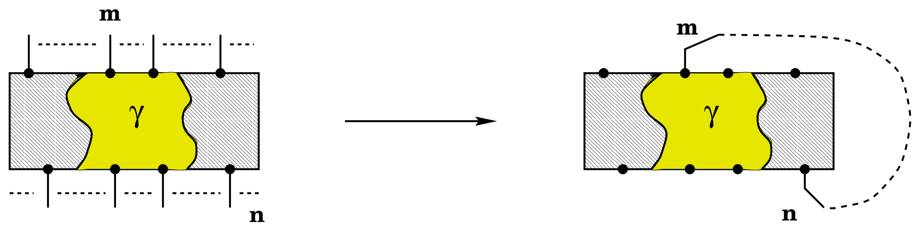


Figure 10: Definition of index j.

Definition 2. *If there exists a pair of external lines m, n such that $c_{G_{mn}}(\gamma) > c_G(\gamma)$, then $j(\gamma) = 1$. Otherwise, $j(\gamma) = 0$.*

From the definition of the index j given above it is easy to see that if $\gamma \subseteq \gamma'$ and $j(\gamma) = 1$, then $j(\gamma') = 1$.

Remark. The meaning of j is the following. Let $\partial\Sigma(G)$ be the boundary of $\Sigma(G)$. If the number of components of $\partial\Sigma(G)$ is greater than one, then $j(G) = 1$. Otherwise, $j(G) = 0$. The condition $j(\gamma) = 1$ for a subgraph $\gamma \subset G$ means that there is at least one hole inside and one hole outside of γ (see figures 2 and 5.).

Definition 3. *The sets $K_1(G)$ and $K_2(G)$ of the lines of a graph G are defined as*

$$K_1(G) = \{l \in G | L(G - l) = L(G)\} \tag{3.1}$$

and

$$K_2(G) = \{l \in G | g(G - l) < g(G)\} \tag{3.2}$$

Remark. $K_1(G) \cap K_2(G) = \emptyset$.

We now give the definition of exceptional external momenta of a 1PI non-commutative graph G. Let $\mathcal{E}(G)$ be the set of external lines of graph G. For a given subgraph $\gamma \subseteq G$, we can decompose $\mathcal{E}(G)$ into the disjoint subsets $\mathcal{E}_s(\gamma)$, $\quad s = 1, \ldots, n(\gamma)$, with the following properties:

- For any s we have $c_{G_{ij}}(\gamma) = c_G(\gamma)$, $\forall i, j \in \mathcal{E}_s(\gamma)$.
- For any s_1 and s_2 such that $s_1 \neq s_2$, we have $c_{G_{ij}}(\gamma) > c_G(\gamma)$, $\forall i \in \mathcal{E}_{s_1}(\gamma)$, $\forall j \in \mathcal{E}_{s_2}(\gamma)$.

The existence and uniqueness of the decomposition

$$\mathcal{E}(G) = \bigcup_{s=1}^{n(\gamma)} \mathcal{E}_s(\gamma) \tag{3.3}$$

for a given γ is proven in lemma 6.4. The meaning of this decomposition is the following. $n(G)$ is simply the number of components of $\partial\Sigma(G)$. In other words it is the number of holes of $\Sigma(G)$. The meaning of $n(\gamma)$ for the subgraph γ can be understood from the figure 5.

Definition 4. *The external momenta $\{p_i\}$ of a 1PI graph G are called exceptional if for at least one $\gamma \subseteq G$ with $n(\gamma) = 2$ in Eq. (3.3) we have*

$$\sum_{i \in \mathcal{E}_1} p_i = \sum_{i \in \mathcal{E}_2} p_i = 0 \tag{3.4}$$

Remark. The meaning of the exceptional momenta is the following. Consider the subgraph $\gamma \subset G$ in figure 5. It separates the holes of $\Sigma(G)$ into two sets. The external momenta are exceptional if the sum of momenta flowing into the holes inside γ (or equivalently, outside of γ) is zero. For the momenta to be exceptional, it is not required to have the momentum flowing into each hole to be zero separately.

Let us now give a generalization of the usual definition of the superficial degree of divergence to the non-commutative theories. We will consider the class of non-commutative field theories with massive propagators. Examples of this class are massive scalar theories with arbitrary couplings (with or without derivatives) and massive spinor theories. The factor in the Feynman integral associated with a line l in a Feynman graph of such theories is of the form

$$\frac{P(q_l)}{q_l^2 + m_l^2}$$

where $P(q_l)$ is a polynomial of degree $\mathrm{deg}(l)$ in the momentum q_l associated with the line l. Let us define an index $\mathrm{ind}_{K_2}(l)$ associated with a line $l \in G$ as

$$\mathrm{ind}_{K_2}(l) = \begin{cases} 1 & \text{if } l \notin K_2 \\ 0 & \text{if } l \in K_2 \end{cases} \tag{3.5}$$

Definition 5. *For an arbitrary subgraph $\gamma \subseteq G$ of a graph G, the degree of divergence $\omega(\gamma)$ is defined as*

$$\omega(\gamma) = dL(\gamma) - 2I(\gamma) + \sum_{l \in \gamma} \mathrm{ind}_{K_2}(l)\mathrm{deg}(l) \tag{3.6}$$

where $L(\gamma)$ and $I(\gamma)$ are the number of independent loops and internal lines of γ, respectively.

The convergence theorem (Theorem 5) can be stated as follows.

Let I_G be the Feynman integral for a 1PI graph G in a field theory over non-commutative $\mathbb{R}^d$ with the massive propagators. There are three cases:

(I) If $j(G) = 0$, then I_G is convergent if $\omega(\gamma) - c_G(\gamma)d < 0$ for all $\gamma \subseteq G$.

(II) If $j(G) = 1$ and the external momenta are non-exceptional, then I_G is convergent if for any subgraph $\gamma \subseteq G$ at least one of the following conditions is satisfied:(1) $\omega(\gamma) - c_G(\gamma)d < 0$, (2) $j(\gamma) = 1$.

(III) If $j(G) = 1$ and the external momenta are exceptional, then I_G is convergent if $\omega(\gamma) - c_G(\gamma)d < 0$ for all $\gamma \subseteq G$.

When $\mathrm{deg}(l) = 0, \forall l \in G$, i.e. for the theories with non-derivative couplings, the above theorem reduces to the one we conjectured in ref.[4].

Let us give a simple example illustrating a peculiar convergence property of diagrams with the derivative couplings related to the definition Eq. (3.6) of ω. Consider the graph G in figure 11. Suppose that there is a factor of p_2^n in the numerator of the Feynman integrand for G. Then

$$I_G = \text{const} \int_0^\infty d\alpha_1 d\alpha_2 \frac{\exp(-\alpha_1 m_1^2 - \alpha_2 m_2^2)}{[\alpha_1\alpha_2 + \theta^2]^{\frac{d}{2}}} \frac{\partial^n}{\partial q^n} \exp[-\frac{\alpha_1 q^2}{\alpha_1\alpha_2 + \theta^2}]\Big|_{q=0} \tag{3.7}$$

(For simplicity we assumed that the absolute values of the eigenvalues of Θ are all equal). It is easy to see that this integral is convergent for any n. This property is directly related to the fact that line 2 of G belongs to the set $K_2(G)$ (see definition 3).

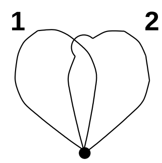


Figure 11: Illustration for a peculiar example.

4 Renormalization

4.1 General discussion.

Let us recall the basic idea behind the subtraction formula of ref.[4]. The vertices in the non-commutative Feynman graphs come from the interaction term $\int \mathcal{L}_{int}$. The interaction term has only cyclic symmetry. Thus one may think of an n-vertex as a rigid ribbon vertex. The topology of a Feynman graph in NQFT arises from *the way* the legs of the n-vertices are *joined* together by the propagators. Locally around the center of a vertex the graph is planar. Thus the Lagrangian is capable of generating only planar vertices.

To illustrate the idea, consider the $U(N)$ real scalar QFT in $d = 4$ with the tree level Lagrangian

$$\mathcal{L} = \text{Tr } [(\partial_\mu\phi)^2 + m^2\phi^2 + g\ \phi^4] \tag{4.1}$$

The Lagrangian Eq. (4.1) is invariant under the global symmetry $\phi \to U^{-1}\phi U$. It is clear that Eq. (4.1) is not the most general $U(N)$ invariant Lagrangian. The quantum corrections will induce the terms of the following forms.

• The overall divergent diagrams of the type "sphere with two holes" will induce the terms $(\text{Tr } \partial_\mu\phi)^2$, $(\text{Tr } \phi)^2$, $(\text{Tr } \phi)(\text{Tr } \phi^3)$ and $(\text{Tr } \phi^2)^2$.

• The overall divergent diagrams of the type "sphere with three holes" will induce the term $(\text{Tr } \phi)^2(\text{Tr } \phi^2)$.

• The overall divergent diagrams of the type "sphere with four holes" will induce the term $(\mathrm{Tr}\ \phi)^4$.

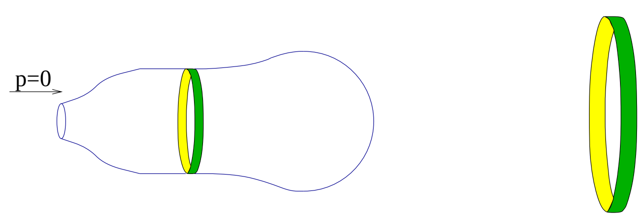
Thus we see that Tr ϕ^4 is capable of subtracting the divergences only from the graphs of the type "sphere with one hole" (=planar graphs). At first sight it seems that the same is true for $\int \phi \star \phi \star \phi \star \phi$ of the NQFT, since $\int$ can be thought of as a trace.

The subtraction of the planar divergences which occur in a non-commutative graph is most natural in BPHZ approach. In the latter approach one does not introduce a regulator, but rather deals directly with the integrand of the Feynman integral. One subtracts the first few terms of the Taylor series of the integrand as a function the external momenta. The topology of a non-commutative graph arises from the phase factors coming from the interaction vertices. Thus to keep the topology of the graph intact, one should not act with the Taylor expansion operator on the phase factors.

Thus, if the planar subgraphs of a non-commutative graph G are the only subgraphs which violate the conditions of the convergence theorem, then these divergences can be subtracted using the counter-terms of the form which occur in the original Lagrangian. This class of graphs was called class Ω_d in ref.[4]. Here we will simply call it class $\mathbf{\Omega}$, keeping in mind that $\mathbf{\Omega}$ refers to a particular theory in a particular dimension.

Note that a planar graph (=sphere with one hole) may contain a divergent subgraph which is a sphere with more than one hole. Consider the sphere with a hole shown in figure 12(a). Since the total external momentum flowing into the hole is zero, the momentum flowing across the ring subgraph (=sphere with two holes) is zero. Thus the phase factors will not regulate this ring subgraph. More technically, this follows from the fact that $c(\gamma) = 0$ for the ring subgraph γ of the sphere with a hole.

Consider ϕ^4 theory in $d = 4$. The graph in figure 13(a) contains a subgraph of the type "sphere with two holes" figure 13(b).[5]



(a) Sphere with one hole (b) Sphere with two holes

Figure 12: A baloon.

Let us analyze the sunset diagrams in figure 15. The planar sunset diagram figure 15(a) is overall divergent and contains three divergent subgraphs: 1-2, 1-3 and 2-3. The

[5]This particular graph was given in ref.[6].

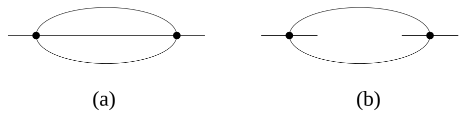


Figure 13: A non-planar subgraph of a planar graph.

overall divergence as well as the divergences of 1-2 and 2-3 can be subtracted using the planar vertices (see figure 16(a)). The basic and the counter-term vertices

$$g\ \phi \star \phi \star \phi \star \phi, \qquad \delta g(\Lambda)\ \phi \star \phi \star \phi \star \phi \tag{4.2}$$

are shown in figure 14. But the subgraph 1-3 seems to be quite problematic because there is no corresponding counter-term graph. By analogy with the commutative $U(N)$ ϕ^4 theory one expects counter term of the type (Tr ϕ^2)^2. Under the identification $\int d^4x \leftrightarrow$ Tr we seem to need

$$\left(\int d^4x\phi^2\right)^2 \tag{4.3}$$

counter term, which is catastrophic. Ref.[6] worked with the single line notation for the Feynman graphs and used the cosines for the interaction vertices to show that ϕ^4 NQFT is 2-loop renormalizable. So what is wrong with our argument regarding the planar sunset diagram? The resolution of this puzzle will be given shortly.

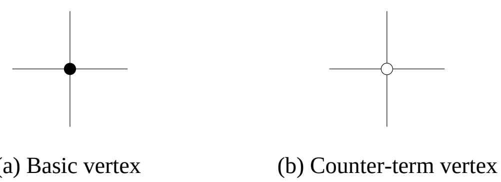


Figure 14: Vertices

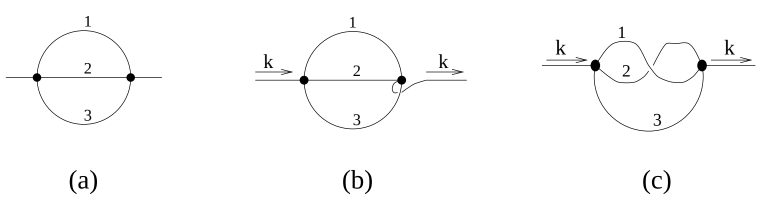


Figure 15: Sunset diagrams.

Consider now the non-planar sunset diagram in figure 15(b). From the convergence theorem it follows that for $k \neq 0$ the non-planar sunset diagram figure 15(b) is overall convergent and the only divergent subgraph is 1-2. The whole graph and the subgraphs

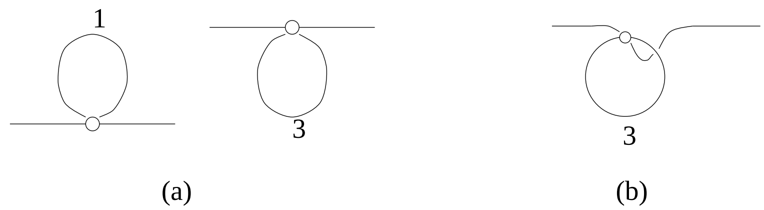


Figure 16: Counter-term graphs.

1-3 and 2-3 have $j(\gamma) = 1$. The corresponding counter-term graph is shown in figure 16(b).

The non-planar sunset diagram figure 15(c) is convergent for any k because all subgraphs satisfy the condition $\omega(\gamma) - 4c(\gamma) < 0$.

Now let us return to our puzzle. In the dimensional regularization the divergent part of the Feynman integral for the planar sunset diagram is of the form

$$c_1 \, \frac{m^2}{\epsilon^2} + c_2 \, \frac{m^2}{\epsilon} + c_3 \, \frac{m^2}{\epsilon} \ln \frac{m^2}{\mu^2} + c_4 \, \frac{k^2}{\epsilon} \tag{4.4}$$

where c_1, c_2, c_3 and c_4 are some numerical constants and the overall factor g^2 is omitted. In the commutative ϕ^4 theory one subtracts the divergences from the subgraphs 1-2, 1-3 and 2-3 by a single counter term graph in figure 16(a) which comes with the factor 3:

$$-3 \left(c_5 \frac{m^2}{\epsilon^2} + c_6 \, \frac{m^2}{\epsilon} + \frac{c_3}{3} \, \frac{m^2}{\epsilon} \ln \frac{m^2}{\mu^2} \right) \tag{4.5}$$

where c_5 and c_6 are some numerical constants. The remaining divergence

$$(c_1 - 3c_5) \, \frac{m^2}{\epsilon^2} + (c_2 - 3c_6) \, \frac{m^2}{\epsilon} + c_4 \, \frac{k^2}{\epsilon} \tag{4.6}$$

can be absorbed in the 2-loop wave function and mass renormalization.

The story is different in the non-commutative ϕ^4 theory. There are only two counter-term graphs figure 16(a) each coming with the factor one. Instead of Eq. (4.6) one has

$$(c_1 - 2c_5) \, \frac{m^2}{\epsilon^2} + (c_2 - 2c_6) \, \frac{m^2}{\epsilon} + \frac{c_3}{3} \, \frac{m^2}{\epsilon} \ln \frac{m^2}{\mu^2} + c_4 \, \frac{k^2}{\epsilon} \tag{4.7}$$

But the extra divergence is independent of k and can be absorbed into the renormalization of mass. Thus our puzzle is resolved.

In addition to the graphs of type $\mathbf{\Omega}$, there are two other types of graphs: **Com** (commutant) and **Rings**. The definition of **Com** follows.

Definition 6. *A Feynman graph in a NQFT is from the class* **Com** *if it contains subgraphs with $\omega \geq 0$ which wrap homologically trivial, but topologically non-trivial cycles (e.g. cycle C in figure 1).*

Consider a subgraph γ which wraps the cycle C in figure 1(b). In ϕ^4 theory in $d = 4$ we have $\omega(G) = 4 - E$ for any 1PI graph G with E external lines. We are interested in the subgraphs with $\omega \geq 0$. Thus for the graph in figure 1(b) to be 1PI, the subgraph γ has to be of the form shown in figure 7. In the ϕ^4 NQFT the graph in figure 7 will be overall convergent as long as $p_1 + p_2 \neq 0$. But if the graph 7 wraps the C cycle in figure 1(b), the momentum conservation will enforce $p_1 + p_2 = 0$. Define $k = p_1 = -p_2$. Figure 17 illustrates the shrinking (=UV divergence) of the subgraph wrapping the commutant. From the figure 17 it is easy to see that the corresponding divergent piece of the Feynman integral will be independent of the momentum k. Thus we can absorb the divergence into the mass renormalization as in figure 18. The basic and counter-term graphs are illustrated in figure 19. The details of this procedure will be discussed in section 4.4 on the example of ϕ^6 NQFT in $d = 2$.

Remark. Let us define the two-two rings in ϕ^4 NQFT to be the 1PI graphs of the type "sphere with two holes" with two external lines on one hole and two on the other hole (see figure 20). It is not difficult to see that the boxes which enclose the subgraphs containing the two-two rings are either disjoint or nested (see figure 21)[6]. For example, the subtraction of the graph in figure 21 goes as follows. All planar divergent subgraphs are subtracted first using the recursive formula of ref.[4]. Since

$$\text{box } 1 \subset \text{box } 2 \subset \text{box } 5 \qquad \text{and} \qquad \text{box } 3 \subset \text{box } 4 \subset \text{box } 5,$$

and box 2 and box 4 do not overlap, box 1 and box 3 are subtracted first and then box 2 and box 4, and finally box 5.

Note that in ϕ^4 theory 1PI graphs with more than four external lines have $\omega < 0$. This means that spheres with two holes and six external lines (four lines on one boundary and two lines on the other boundary) are overall convergent. Thus the counter-terms for the graphs of the type in figure 18 but with four external lines are not needed. We thus conclude that δg is independent of θ.

From the above discussion it follows that the β-function

$$\beta = \mu \frac{dg(\mu)}{d\mu} \tag{4.8}$$

will be independent of θ whereas the renormalization group coefficient [7]

$$\gamma_m = -m^{-2} \mu \frac{dm^2(\mu)}{d\mu} \tag{4.9}$$

will be a non-trivial function of θ.

We have seen that spheres with two holes are overall convergent at non-exceptional external momenta, but can diverge inside a bigger graph. One may wonder if the same is

[6]More precisely, the boxes enclose the two-two ring and the "dead end" part of the graph as can be seen from the figure 21.

[7]In the commutative QFT in the minimal subtraction scheme, β and γ_m depend only on g.

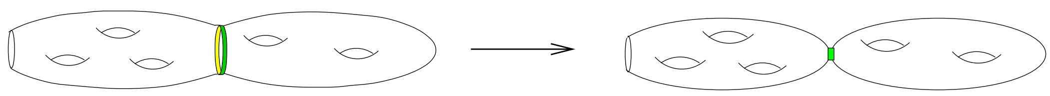


Figure 17: Shrinking (=UV diverging) commutant.

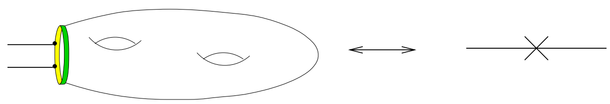
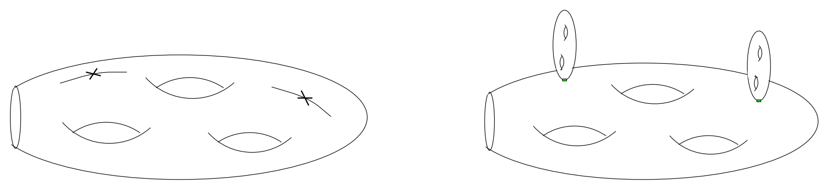


Figure 18: Counter-term



(a) Counter-term graph (b) Basic graph

Figure 19: Counter-term and basic graphs.

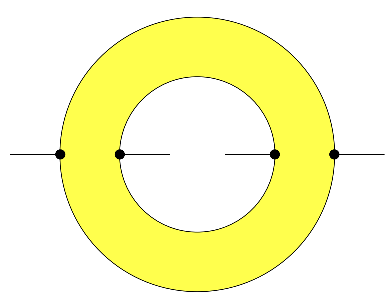


Figure 20: Two-two ring.

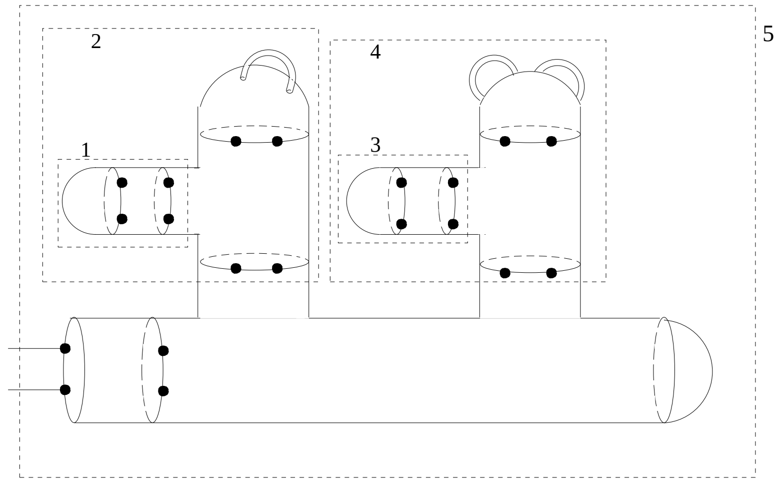


Figure 21: Nested and disjoint boxes.

true for the graphs of the type "sphere with $n > 2$ holes". Consider the sphere with three holes ("pants") and four external lines shown in figure 22. If it is part of a bigger 1PI graph G and the external lines 1 and 2 are internal lines of G, then it has to be joined to the rest of the diagram as in figure 23. But then there will be virtual momentum q circling as in figure 23. The dotted boxes in figure 23 indicate the subgraphs of the type sphere with two holes. Thus in an obvious sense the case of "spheres with $n > 2$ holes" reduces to that of "spheres with two holes".

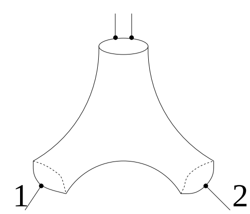


Figure 22: Pants

We have been discussing the spheres with two holes which wrap commutants of a very specific type, namely, those which "separate the handles" (see, for example, the commutant C in figure 1(b)). There are actually some other types of commutants as, for example, in figure 8. Should this type of commutants be considered in the divergence analysis? The answer is no. For the theories of interest, ω for such a subgraph γ will be negative because γ will have too many external lines (see figure 9(b)).

We now define a new class of graphs.

Definition 7. *A Feynman graph in a NQFT is from the class* **Rings** *if it contains subgraphs which wrap a homologically non-trivial cycle and satisfy the condition* $\omega(\gamma) - d < 0$,

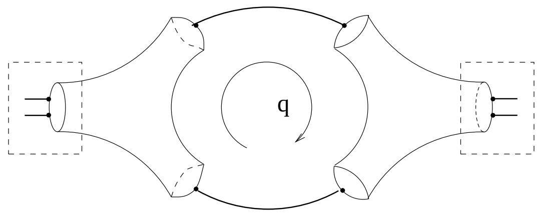


Figure 23: A pair of pants

but the disjoint union of these subgraphs violate the condition $\omega(\gamma) - d < 0$.

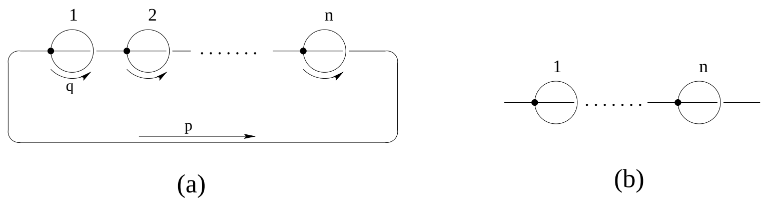


Figure 24: A chain of rings in ϕ^4 theory.

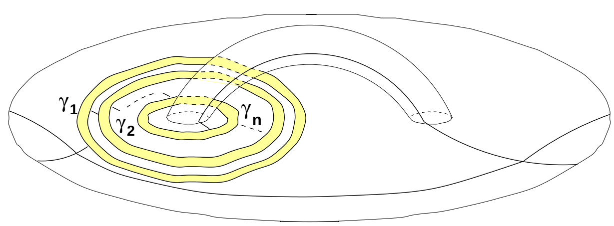


Figure 25: A diagram from the class **Rings** in ϕ^4, $d = 4$ theory.

Remark. A particular example of graph from **Rings** is shown in figure 24(a). Ref.[5] first noted that the latter graph for $n \geq 2$ is divergent in $d = 4$ by the explicit calculation. The Feynman multiple integral $\int d^4q d^4p \cdots$ is divergent. If the integration over q is performed first, then the integral $\int d^4p$ diverges at $p = 0$ (IR divergence). If the integration over p is performed first, then the integral $\int d^4q$ diverges at $q = \infty$ (UV divergence). The general graph in figure 25 is from the class **Rings**.

Now let us ask the following question. Let G be a 1PI graph in ϕ^4 NQFT in $d = 4$. Is it possible to have a 1PI subgraph $\gamma \subset G$ with $c_G(\gamma) \geq 1$ which would violate the convergence condition $\omega(\gamma) - c_G(\gamma)d < 0$?

The answer is no. The reason is simple: for any 1PI graph which is not a vacuum bubble in this theory we have $\omega(\gamma) < d = 4$.

We thus conclude that any subgraph γ with $c_G(\gamma) \geq 1$ which violates the condition $\omega(\gamma) - c_G(\gamma)d < 0$ has to be a disjoint union of 1PI graphs with $c \geq 1$. This is true for any theory in which $\omega(G) < d$ for any 1PI graph G with external lines. We now show that only $c = 1$ disjoint graphs can violate the convergence condition. This follows from the following obvious statement:

If

(1) γ_1 and γ_2 are 1PI subgraphs of G, and

(2) γ_1 and γ_2 wrap the same cycles, and $c_G(\gamma_1) = c_G(\gamma_2) > 1$,

then $\gamma_1 \cup \gamma_2$ is 1PI. Thus $\gamma_1 \cup \gamma_2$ is connected.

Figure 26 illustrates a generic structure of 1PI **Rings** graphs in ϕ^4 NQFT. Note that the rings do not overlap[8]. The sub-surfaces Σ and Σ' do not contain any ring. In general, there will be n sub-surfaces joined together in all possible ways by the "tubes" containing rings.

Ref.[5] suggests the following procedure to deal with the divergence of the graph 24(a) with $n \geq 2$. The connected diagram shown in figure 24(b) is convergent for the non-zero external momentum k. It is divergent at $k = 0$. A power counting argument shows that for sufficiently small k the $n = 1$ approximation is not adequate and $n \geq 2$ diagrams become important. Thus one has to sum over n. The resulting integral over k is convergent at $k = 0$ (see ref.[5] for the details).

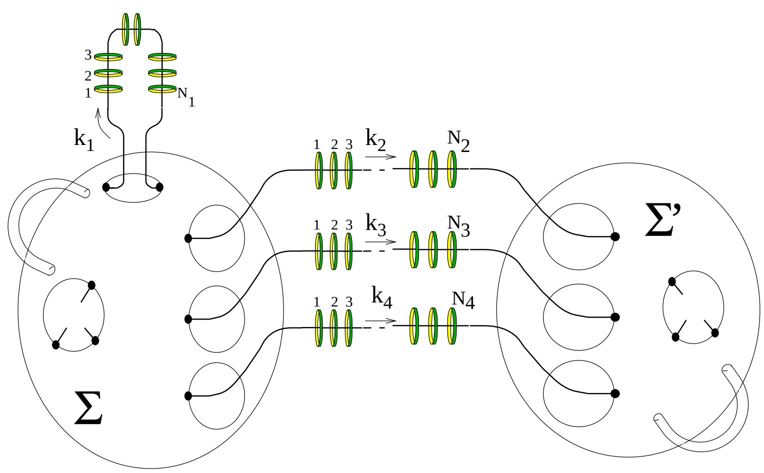


Figure 26: A generic structure of **Rings** graphs.

Now let us denote by $\Pi_L(k)$ the sum of contributions (after the subtraction of the planar subdivergences) of up to L-loop 1PI diagrams on the r.h.s. of the equation in

[8]A ring has only two external vertices which are reserved for the joining to the rest of the rings wrapping a given cycle. Thus a ring wrapping a given cycle cannot overlap a ring wrapping a different cycle. Hence the structure shown in figure 26.

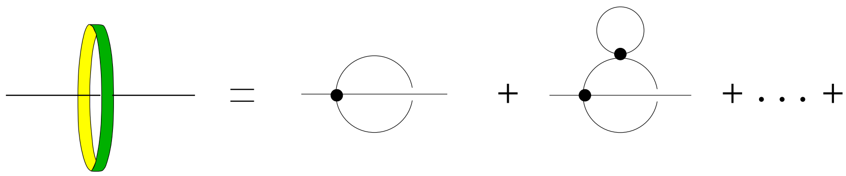


Figure 27: A sum of 1PI rings.

figure 27. It is not difficult to see that the loop corrections within an individual ring are not important at small g. Thus in computing a Green's function in perturbation theory for a given accuracy, it is enough to have $L < L_{max}$ for some given L_{max}. The surfaces Σ and Σ' in figure 26 also have the number of loops bounded from above by L_{max}.

Now consider the graph in figure 26. The non-overlapping structure of the rings allows to re-group the set of graphs as in figure 26, where the ring is given by figure 27. Denoting by $k_1, \dots, k_4$ the momenta flowing along the cycles to which the rings are attached, we have, after the summation $\sum_{N_1,N_2,N_3,N_4=1}^{\infty}$, the product

$$\frac{\Pi_L(k_1)}{(k_1^2+m^2)(k_1^2+m^2-\Pi_L(k_1))} \cdots \frac{\Pi_L(k_4)}{(k_4^2+m^2)(k_4^2+m^2-\Pi_L(k_4))} \tag{4.10}$$

In the limit $k \to 0$ we have $\Pi_L(k) \to \infty$ and thus Eq. (4.10) reduces to

$$\frac{1}{(k_1^2+m^2)(k_2^2+m^2)(k_3^2+m^2)(k_4^2+m^2)} \tag{4.11}$$

Suppose that the sub-surfaces Σ and Σ' in figure 26 are convergent. Since Σ and Σ' do not contain rings, this can always be achieved by the subtraction procedure discussed earlier. Now we join Σ and Σ' using the resummed rings. Since $\Pi_L(k)$ is exponentially vanishing at $k \to \infty$, the integrals

$$\int d^4k_1 \int d^4k_2 \int d^4k_3 \int d^4k_4 \cdots \tag{4.12}$$

are convergent at $k_i \to \infty$.

Thus only the convergence at $k = 0$ is questionable. From Eq. (4.11) it is clear that the resummed rings are harmless at $k \to 0$. Thus the only possible divergence may come from the surfaces Σ and Σ'. Recall that Σ and Σ' may diverge at the exceptional external momenta. When we join Σ and Σ' using the resummed rings, we necessarily make some of the external momenta (which actually become internal lines in the resulting graph) exceptional. But these momenta are integrated over (we are talking about the momenta k_1,...,k_4 in figure 26). Since Σ and Σ' do not contains rings, we conclude that the integral Eq. (4.12) is convergent at $k_i \to 0$. Thus at least at a heuristic level it is clear that the divergences of **Rings** graphs can be dealt with.

4.2 Wess-Zumino model

In ref.[4] we suggested that the non-commutative version of the Wess-Zumino model (in $d = 4$) is renormalizable to all orders in the perturbation theory. The argument given in ref.[4] is short and precise: $\omega \leq 0$ for 1PI supergraphs. Thus there are no **Rings** graphs[9].

The argument given in ref.[4] is not complete because

(1) The subtraction procedure in BPHZ scheme necessarily introduces the propagators of the form $P(k)/(k^2 + m^2)$, where $P(k)$ is a polynomial in k. The convergence theorem for the NQFT with the propagators of this form was neither proven nor argued to be true in ref.[4], but it was used in the argument for the renormalizability of Wess-Zumino model.

(2) The convergence theorem was formulated for the ordinary graphs, but used for the supergraphs.

(3) The **Rings** graphs were shown to be absent simply because there are at most the logarithmic divergences.[10]. But it was not shown that **Com** graphs are absent.

Point (1) is dealt with in section 7. Point (2) can be dealt with as follows. The star product in the Wess-Zumino action does not depend on the Grassmann variables θ and so the super Feynman graphs are the ribbon graphs. As in the commutative case, there will be θ's associated with each vertex. In the language of the Feynman integrals this means that only the phase factors will carry the information about the topology of the graph and these phase factors are the same as in non-commutative ϕ^3 theory. Apart from the phase factors, the rest of the Feynman integrand is identical to that of the commutative Wess-Zumino model. For further discussions of the superfield formalism in SUSY NQFT see ref.[8].

The fact that there are no 1PI **Com** graphs can easily be seen as follows. Let us look at the genus two graph shown in figure 1(b). In order for the graph to be 1PI, any subgraph which wraps the C cycle should have at least two external legs on each hole as in the figure 7. But such graphs have $\omega < 0$. Thus there are no **Com** graphs. Note that by a graph we mean a supergraph.

Thus the divergences from all 1PI supergraphs (with the non-exceptional external momenta) in the Wess-Zumino model can be removed. What about the non-1PI connected graphs? In principle, the non-1PI graphs of the type shown in figure 28(a) are problematic. But in this theory the graphs with one external line are identically zero. Thus the general discussion of this section suggests that the Wess-Zumino model should be renormalizable to all loop orders.[11]

[9] Refs. [5, 9, 10, 12] subsequently studied one-loop diagrams in supersymmetric NQFT and found that IR poles typical of NQFT are absent (more precisely, there are only logarithmic singularities).

[10] This can most easily be seen in the superspace language.

[11] A different approach for the renormalization of non-commutative Wess-Zumino model was suggested in ref.[12].

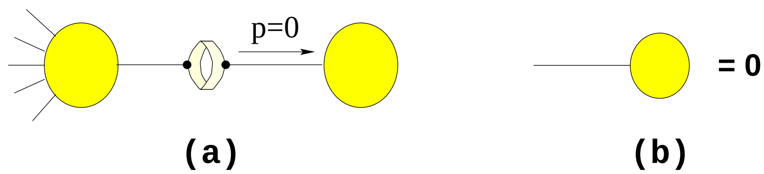


Figure 28: The supergraphs which are identically zero in Wess-Zumino model.

4.3 ϕ^4 in $d = 2$

The corresponding commutative theory is super-renormalizable. The superficial degree of divergence of a 1PI graph with V vertices in this theory is $\omega = 2 - 2V$. So ω is at most 0. There are no 1PI **Rings** and **Com** graphs. Thus the divergences from all 1PI graphs can be removed by the introduction of the planar counter-terms in the Lagrangian.

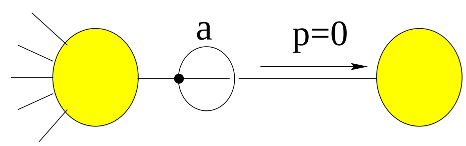


Figure 29: A divergent non-1PI graph.

At first sight the situation seems to be different with the non-1PI graphs. There seems to be a specific type of non-1PI connected graphs which contain the nonplanar divergences. The graph in figure 29 contains a non-planar divergent subgraph a. The momentum flowing across this subgraph is zero by the momentum conservation. But the graphs of the type shown in figure 29 are impossible in ϕ^4 theory because there are no graphs with one external leg in this theory. The latter statement follows from the $\phi \to -\phi$ symmetry of the action or, equivalently, from the relation $E + 2I = 4V$. Thus ϕ^4 theory in $d = 2$ is renormalizable to all orders in the pertubation theory. By this we mean that the divergences from all connected Green's functions at non-exceptional external momenta can be removed in the counter-term approach.

4.4 ϕ^6 in $d = 2$

The quantum corrections to ϕ^6 action will induce ϕ^4 terms. Thus we consider the combined ϕ^6 plus ϕ^4 action in $d = 2 - 2\epsilon$:

$$\begin{aligned} S &= \int d^{2-2\epsilon}x \, [\frac{1}{2}(\partial_\mu \phi)^2 + \frac{1}{2}(m^2 + \delta m^2)\phi^2 + \frac{1}{4}(g_4 + \delta g_4) \, \mu^{2\epsilon} \, (\phi \star \phi \star \phi \star \phi) \\ & \quad + \frac{1}{6} g_6 \, \mu^{4\epsilon} \, (\phi \star \phi \star \phi \star \phi \star \phi \star \phi)] \end{aligned} \tag{4.13}$$

(a) One loop divergent graphs (b) Counter-term vertices

Figure 30: One loop divergent graphs in ϕ^6 theory in $d = 2$.

For convenience let us define

$$\begin{aligned} \mu^{2\epsilon} \int \frac{d^{2-2\epsilon}q}{(2\pi)^{2-2\epsilon}} \frac{1}{(q^2+m^2)} &= \frac{1}{4\pi} \left[\frac{1}{\epsilon} + \psi(1) - \ln\frac{m^2}{4\pi\mu^2} + \mathcal{O}(\epsilon)\right] \\ &\equiv \frac{c_0}{\epsilon} + c_1 + \epsilon\, c_2 + \cdots \end{aligned} \tag{4.14}$$

We will show that the theory with the action Eq. (4.13) is finite in the $\epsilon \to 0$ limit, provided that we choose appropriate δm^2 and δg_4:

$$\delta g_4 \equiv -\frac{4c_0 g_6}{\epsilon} \tag{4.15}$$

and

$$\begin{aligned} \delta m^2 &= -\frac{2c_0 g_4}{\epsilon} + \frac{3c_0^2 g_6}{\epsilon^2} - \frac{2c_0 c_1 g_6}{\epsilon} \\ &\quad + \frac{2c_0 g_4 g_6 m^{-2}}{\epsilon} f(\theta m^2) + (n > 3) \text{ loop corrections} \end{aligned} \tag{4.16}$$

where f is given by

$$\frac{1}{m^2}\, f(\theta m^2) \equiv \int \frac{d^2q}{(2\pi)^2} \frac{d^2p}{(2\pi)^2} \frac{\exp(ip\wedge q)}{(p^2+m^2)^2(q^2+m^2)} \tag{4.17}$$

Note that δm^2 receives contributions from all loop orders.

Figure 31: Subtraction in the commutative theory.

The one loop divergent graphs and the corresponding mass and coupling constant vertices are listed in figure 30. The one loop mass and coupling constant corrections are

$$\delta m^2 = -\frac{2c_0 g_4}{\epsilon} \tag{4.18}$$

and

$$\delta g_4 = -\frac{4c_0 g_6}{\epsilon} \tag{4.19}$$

The one loop coupling constant counter-term δg_4 does not receive corrections from the higher loops.

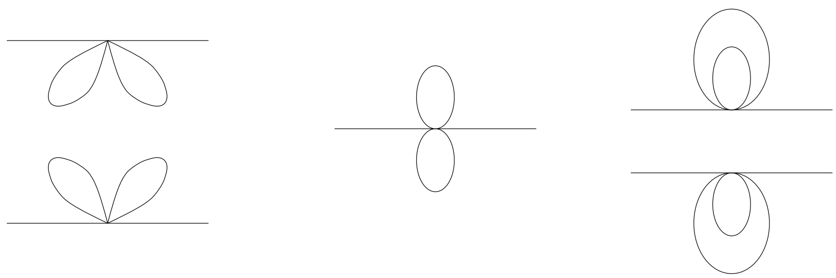


Figure 32: Diagrams contributing to the 2-loop mass renormalization.

Consider the two loop graph in figure 31. In the commutative theory it comes with the factor of 15. The subtractions are done as in figure 31. In the non-commutative theory there are 15 different diagrams each with the factor one. Some of these graphs are shown in figures 32 and 34(a). The counter-term graph for the graph 34(a) is shown in figure 34(b). The graphs in figure 32 are the only graphs which contribute to two loop mass counter-term. Two of the graphs in figure 32 contain divergent non-planar one loop subgraph of the type "sphere with two holes". As explained in section 4.1., the corresponding divergence can be absorbed in the two loop mass counter-term. The sum of the contributions from figures 32 and 33 reads

$$\begin{aligned} & -5g_6(\frac{c_0}{\epsilon} + c_1 + \cdots)^2 - 2\delta g_4(\frac{c_0}{\epsilon} + c_1 + \cdots) \\ & = \frac{3c_0^2 g_6}{\epsilon^2} - \frac{2c_0 c_1 g_6}{\epsilon} + \text{finite} \end{aligned} \tag{4.20}$$

Thus the mass counter-term up to two loops reads

$$\delta m^2 = -\frac{2c_0 g_4}{\epsilon} + \frac{3c_0^2 g_6}{\epsilon^2} - \frac{2c_0 c_1 g_6}{\epsilon} \tag{4.21}$$

Figure 33: Counter-term graphs.

In this theory there are no 1PI **Rings** graphs since $\omega = 2 - 2V$ is at most zero. But there are 1PI **Com** graphs. In order for a graph to wrap a commutant (see cycle C in

Figure 34: Basic and counter-term graph.

figure 1), it should be "a sphere with two holes". It is easy to see that the only such graphs (with four external lines) which are divergent at the vanishing total momentum flowing into each hole are the ones shown in figure 35.

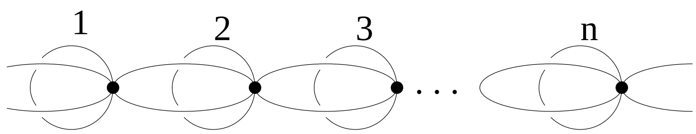


Figure 35: A chain of rings in ϕ^6 theory.

The first non-trivial θ dependent contribution to δm^2 comes from the 3-loop graph in figure 36(a). The Feynman integral for the subgraph in figure 36(b) is given by Eq. (4.17). Thus δm^2 is given up to 3-loops by Eq. (4.16).

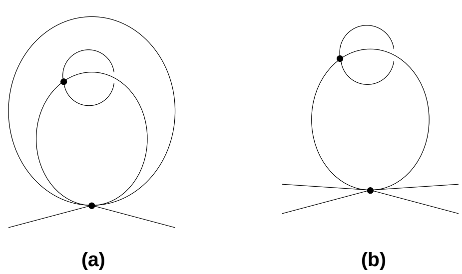


Figure 36: The first non-trivial diagram in ϕ^6 NQFT in $d = 2$.

Now consider the graph in figure 37(a). It contains a divergent non-planar subgraph 37(b). It is easy to see that the divergence is independent of the external momentum. As in section 4.1 the divergence of the graph 37(a) is absorbed in the mass renormalization. If the Feynman integral corresponding to the subgraph shown in figure 37(c) is convergent, the contribution of figure 37(a) to the mass counter-term will be

$$\frac{1}{\epsilon} g_4^{V_4} g_6^{V_6} m^{\omega(G)} F(\theta m^2) \tag{4.22}$$

for a non-trivial function F if the graph 37(c) is non-planar, where V_4 and V_6 are the number of ϕ^4 and ϕ^6 vertices, respectively.

Remark. The analysis of this subsection can readily be extended to ϕ^{2n} NQFT in $d = 2$. It is not difficult to see that the counter-terms δm^2, δg_4, $\delta g_6, \ldots$, δg_{2n-4}, will depend non-trivially on θ.

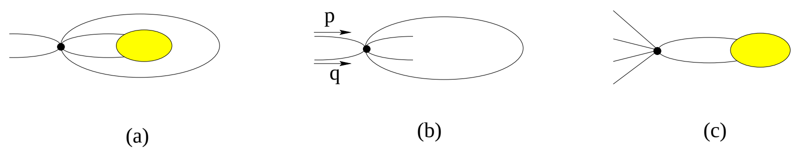


Figure 37: The graphs which contain non-planar subgraphs.

4.5 $\phi^* \star \phi \star \phi^* \star \phi$ in $d = 4$

This theory is interesting because **Rings** graphs are absent for a purely graph-theoretic reason. This should be contrasted with what happens in supersymmetric non-commutative theories in general and non-commutative Wess-Zumino model in particular. In SUSY theories **Rings** graphs are absent because $\omega \leq 0$ for the supergraphs.

A general 1PI graph in this theory can be drawn on a genus g surface with a number of holes. The only difference from a graph in the real ϕ^4 theory is that the lines in the complex ϕ^4 theory carry arrows whereas the lines in the real ϕ^4 theory do not carry them. In other words the 2-surfaces in the complex ϕ^4 theory are *decorated* whereas those in the real ϕ^4 theory are not. Let us consider the piece of a graph in $\phi^* \star \phi \star \phi^* \star \phi$ theory shown in figure 38(a). Associating the clockwise direction of the arrows with the color A and the counter-clockwise direction with the color B, we can equivalently draw the graph as in figure 38(b). Let G be a graph in the real ϕ^4 NQFT and $\Sigma(G)$ the associated surface. Then it is easy to see that G is also a graph in the complex ϕ^4 theory if and only if the plaquettes of $\Sigma(G)$ can be painted in two colors as in figure 38(b). According to a well-known coloring theorem, any planar graph in the real ϕ^4 NQFT admits coloring and thus it is a graph of the complex ϕ^4 theory.

Let $\Sigma(G)$ be the surface associated with a graph G in $\phi^* \star \phi \star \phi^* \star \phi$ theory. As in the figure 39 the total number of external lines attached to each hole of $\Sigma(G)$ is always an even number and the arrows carried by these lines always appear in the alternating order in-out-in-out-... . This statement is evident from the figure 40, where the local structure at the boundary of a hole is depicted.

This means that there are no 1PI graphs of the type "sphere with two holes" and two external lines shown in figure 41 in the $\phi^* \star \phi \star \phi^* \star \phi$ theory. Thus there are no **Rings** graphs.

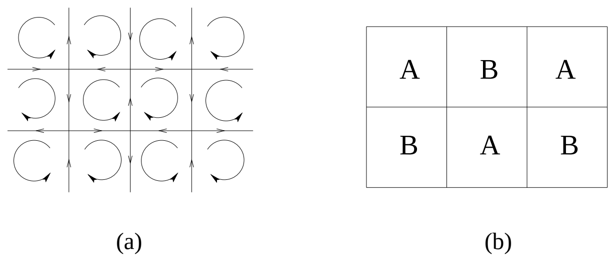


Figure 38: A graph in $\phi^* \star \phi \star \phi^* \star \phi$ theory and two color problem.

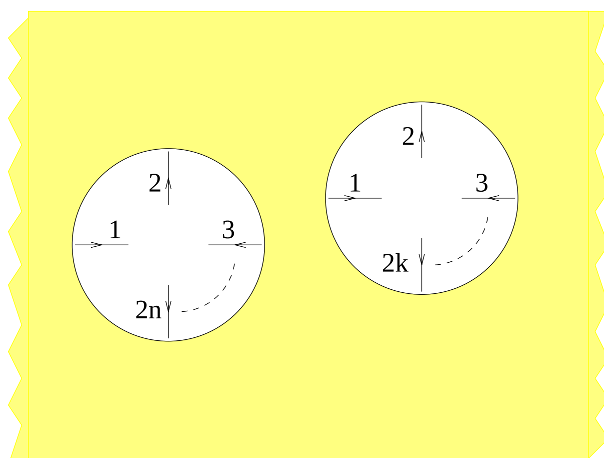


Figure 39: Two holes.

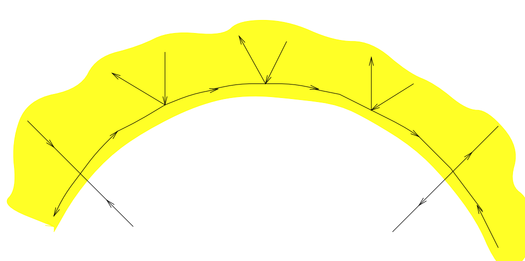


Figure 40: Local structure at the boundary.

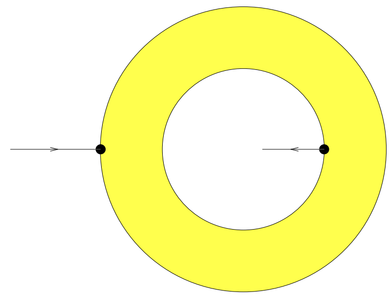


Figure 41: Impossible ring diagram.

It is also clear that the graphs of the type "sphere with two holes" with four external lines shown in figure 7 are permissible in this theory. Thus there will be **Com** graphs in this theory. There will also be the planar graphs which contain divergent non-planar subgraphs of the type shown in figure 7.

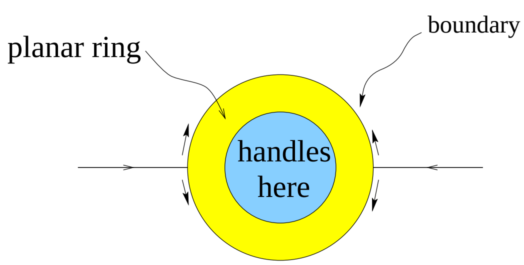


Figure 42: Impossible diagram.

The external lines of a planar 1PI graph with four external lines are necessarily in the order $\phi^* - \phi - \phi^* - \phi$ (see figure 39). Thus the overall divergences from the planar 1PI graphs with four external lines can be removed by the coupling constant counter-term

$$\delta g\ \phi^* \star \phi \star \phi^* \star \phi$$

Let us now consider the counter-term of the type shown in figure 18. From the figures 39 and 42, it is easy to see that ϕ^2 and ϕ^{*2} terms cannot be induced in $\phi^* \star \phi \star \phi^* \star \phi$ theory. Thus the divergences from the diagrams of the type shown in 18 can be subtracted using mass counter-term $\delta m^2 \phi^* \phi$.

It can be shown diagrammatically that the mass counter-term δm^2 is independent of θ up to 3 loop order. The first non-trivial θ dependent contribution to δm^2 comes from the 4 loop diagram shown in figure 43(a). The Feynman integral corresponding to the graph in figure 43(b) is convergent. On dimensional grounds it is of the form $m^2 f(\theta m^2)$, where f is a non-trivial function. Thus the contribution to δm^2 is

$$\frac{\text{const}}{\epsilon}\ m^2 f(\theta m^2) \tag{4.23}$$

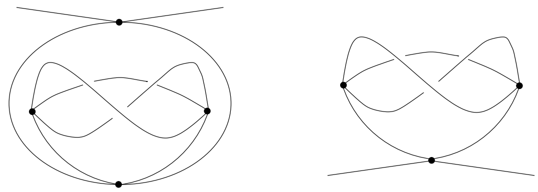


Figure 43: The first non-trivial diagram in $\phi^* \star \phi \star \phi^* \star \phi$ NQFT in $d = 4$.

The coupling constant counter-term δg does not depend on θ as follows from the discussion in section 4.1.

From the general discussion of this section it is quite clear that $\phi^* \star \phi \star \phi^* \star \phi$ theory should be renormalizable to all orders in the perturbation theory.

4.6 A comment on non-commutative super Yang-Mills theories (NSYM).

The bosonic non-commutative YM theory contains **Com** graphs. As we discussed before, in the massive theories the divergences from this type of graphs can readily be removed by the mass renormalization. In the commutative YM theory tadpole type graphs vanish. Recall that in the dimensional regularization

$$\int d^d p (p^2)^\alpha = 0$$

Due to the presence of the scale θ in the non-commutative case, the situation with the tadpoles is somewhat involved. Thus **Com** graphs are quite problematic in the non-commutative YM. A priori there is no reason for the cancellation of divergences in **Com** graphs in non-commutative YM.

Ref.[9] reports that there are quadratic divergences in non-commutative YM. Thus there are **Rings** graphs in this theory.

How about NSYM? In these theories $\omega \leq 0$ and therefore **Rings** graphs are absent. It is easy to see that in general there will be **Com** graphs. But it is likely that $N = 4$ NSYM theory is an exception in this respect and there are no **Com** graphs in this theory. The ref.[9] computes one-loop diagrams in $N = 4$ NSYM and shows that IR singularities are absent. At one loop IR divergences come from the spheres with two holes. In the theories with $N < 4$ SUSY, there are logarithmic IR singularities.

The analysis of divergences given in this paper is based on the convergence theorem for the massive NQFT. The convergence properties of the Feynman integrals in massless NQFT are much more involved (see the first comment in section 8). Thus, NYM theories are expected to have rich divergence structures.

5 Convergence theorem for non-derivative scalar commutative theories

To make the proofs given in sections 6 and 7 easier to follow, we urge the reader to follow through the proof of the theorem for commutative scalar theories given in this section.

In a commutative scalar theory with non-derivative couplings the parametric representation of a Feynman integral for an arbitrary graph G reads[12]

$$I_G(p) = \int_0^\infty d\alpha_1 \cdots d\alpha_I \frac{\exp - \left[\sum \alpha_l m_l^2 + Q(p,\alpha)\right]}{P(G)^{\frac{d}{2}}} \tag{5.1}$$

where Q is a quadratic, positive definite form in the external momenta p and a homogeneous rational fraction of degree 1 in the α. Hence, only the convergence at $\alpha = 0$ is questionable. The exponent being bounded at the origin plays no role. The polynomial $P(G)$ is a sum of monomials of degree L. Let $G_i, \;\; i = 1, \ldots, n$ be 1PI components of the graph G. Then

$$P(G) = \prod_i P(G_i) \tag{5.2}$$

As in ref.[15], let us divide the integration domain into sectors

$$0 \leq \alpha_{\pi_1} \leq \alpha_{\pi_2} \leq \cdots \leq \alpha_{\pi_I} \tag{5.3}$$

where π is a permutation of $(1, 2, \ldots, I)$. We shall prove the convergence of the integral sector by sector. To each sector corresponds a family of nested subsets γ_l of lines of G:

$$\gamma_1 \subset \gamma_2 \subset \cdots \subset \gamma_l \subset \cdots \subset \gamma_I \equiv G \tag{5.4}$$

where γ_l contains the lines pertaining to $(\alpha_{\pi_1}, \ldots, \alpha_{\pi_l})$. In a given sector specified by π, we perform a change of variables

$$\begin{array}{rclllllll}
\alpha_{\pi_1} & = & \beta_1^2 & \beta_2^2 & \cdots & \beta_s^2 & \cdots & \beta_{I-1}^2 & \beta_I^2 \\
\alpha_{\pi_2} & = & & \beta_2^2 & \cdots & \beta_s^2 & \cdots & \beta_{I-1}^2 & \beta_I^2 \\
 & \vdots & & & & & & & \\
\alpha_{\pi_s} & = & & & & \beta_s^2 & \cdots & \beta_{I-1}^2 & \beta_I^2 \\
 & \vdots & & & & & & & \\
\alpha_{\pi_{I-1}} & = & & & & & & \beta_{I-1}^2 & \beta_I^2 \\
\alpha_{\pi_I} & = & & & & & & & \beta_I^2
\end{array} \tag{5.5}$$

the jacobian of which is

$$\frac{D(\alpha_{\pi_1}, \ldots, \alpha_{\pi_I})}{D(\beta_1, \ldots, \beta_I)} = 2^I \beta_1 \beta_2^3 \cdots \beta_I^{2I-1} \tag{5.6}$$

[12] In the commutative case one usually considers Eq. (5.1) only for 1PI graphs. For the purpose of the proof of the convergence theorem in the non-commutative case, we will find it very useful to consider arbitrary disconnected graphs. It is easy to see that Eq. (5.1) holds for general disconnected graphs.

In these β variables the integration domain reads

$$0 \le \beta_I \le \infty \quad \text{and} \quad 0 \le \beta_l \le 1 \quad \text{for} \quad 1 \le l \le I-1 \tag{5.7}$$

Using the well-known graph-theoretic formula for $P(G)$ one may prove the following relation [15]

$$P(G) = \beta_1^{2L(\gamma_1)} \beta_2^{2L(\gamma_2)} \cdots \beta_I^{2L(\gamma_I)} [1 + \mathcal{O}(\beta^2)] \tag{5.8}$$

The integrand is majorized by the factor

$$\frac{\beta_1 \beta_2^3 \cdots \beta_I^{2I-1}}{\beta_1^{dL(\gamma_1)} \beta_2^{dL(\gamma_2)} \cdots \beta_I^{dL(\gamma_I)}} = \prod_{l=1}^{I} \beta_l^{-dL(\gamma_l)+2l-1} = \prod_{l=1}^{I} \beta_l^{-\omega_l - 1} \tag{5.9}$$

and since $\omega_l < 0$, the integral $\int_0 \prod d\beta_l \beta_l^{-\omega_l-1}$ is convergent at the origin.

For the purpose of generalization to non-commutative theories, we will give an alternative proof of the relation Eq. (5.8) which does not require the use of the graph-theoretic formula. The only properties of $P(G)$ which we will need for the proof are

- $P(G)$ is a homogeneous polynomial in α of degree $L(G)$
- $$P(G) = \begin{cases} P(G-l) & \text{if } L(G-l) = L(G) \\ \alpha_l P(G-l) + X & \text{if } L(G-l) = L(G) - 1 \end{cases} \tag{5.10}$$

where $G - l$ is the graph G with the line l removed, X is a sum of monomials of degree L in α and it does not depend on α_l. For a 1PI graph G, one can have only the case $L(G-l) = L(G) - 1$ in Eq. (5.10).

It is easy to check that Eq. (5.8) holds for all one loop graphs. Assuming that Eq. (5.8) holds for all of the graphs which have at most L loops, let us prove that Eq. (5.8) holds for all $L+1$ loop graphs. Let G be a general $L+1$ loop graph with I lines. Let us perform a change of variables Eq. (5.5) in $P(G)$.

Let $n \ge 1$ be an integer such that the removal of any one of the lines $\pi_I, \pi_{I-1}, \ldots, \pi_{n+1}$ from the graph G does not change the number of loops and the removal of line π_n from G changes the number of loops. That is

$$L(G - \pi_n) = L(G) - 1$$

$$L(G - \pi_i) = L(G), \qquad n+1 \le i \le I \tag{5.11}$$

From Eq. (5.10) it follows that $P(G)$ does not depend on $\alpha_{\pi_I}, \alpha_{\pi_{I-1}}, \ldots, \alpha_{\pi_{n+1}}$. Since $P(G)$ is a homogeneous polynomial of order $L(G)$ in α, we have

$$P(G) = (\beta_n \beta_{n+1} \cdots \beta_I)^{2L(G)} P(G|\beta_n = 1, \ldots, \beta_I = 1) \tag{5.12}$$

From Eq. (5.10) we have

$$\begin{aligned} &P(G|\beta_n = 1, \ldots, \beta_I = 1) \\ &\quad = \beta_{n-1}^{2L(G-\pi_n)} [P(G - \pi_n|\beta_{n-1} = 1, \ldots, \beta_I = 1) + \mathcal{O}(\beta_{n-1}^2)] \end{aligned} \tag{5.13}$$

Now since $G - \pi_n$ is an L loop graph, we have by inductive assumption and the fact that $P(G - \pi_n)$ does not depend on $\alpha_{\pi_{n+1}}, \ldots, \alpha_{\pi_I}$, [13]

$$P(G - \pi_n | \beta_n = 1, \ldots, \beta_I = 1) = \left(\prod_{i=1}^{n-1} \beta_i^{2L(\gamma_i)} \right) [1 + \mathcal{O}(\beta^2)] \tag{5.14}$$

Using Eq. (5.13), Eq. (5.14) and relations

$$L(G) \equiv L(\gamma_I) = L(\gamma_{I-1}) = \cdots = L(\gamma_n), \qquad L(G - \pi_n) = L(\gamma_{n-1}) \tag{5.15}$$

we find Eq. (5.8).

6 Convergence theorem for non-commutative theories–Non-derivative case

6.1 Outline of the proof

Let us recall the parametric representation of a Feynman integral for a graph G in the scalar field theory over non-commutative $\mathbb{R}^d$ without derivative couplings[4]. It reads

$$I_G(p) = \int_0^\infty d\alpha_1 \cdots d\alpha_I \frac{\exp - \left[\sum \alpha_l m_l^2 + Q(p, \alpha, \Theta) + i\tilde{Q}(p, \alpha, \Theta)\right]}{\prod_{i=1}^{d/2} P(G, \theta_i)} \tag{6.1}$$

where

$$\Theta = \begin{pmatrix} 0 & -\theta_1 & & & \\ \theta_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & -\theta_{\frac{d}{2}} \\ & & & \theta_{\frac{d}{2}} & 0 \end{pmatrix}, \tag{6.2}$$

is $d \times d$ anti-symmetric matrix in the Jordan form,

$$P(G, \theta) = \det \mathcal{A} \det \mathcal{B}$$

$$\mathcal{A}_{mn}(\theta) \equiv \alpha_m \delta_{mn} - \theta I_{mn}, \qquad \mathcal{B}_{v\tilde{v}}(\theta) = \epsilon_{vm} (\mathcal{A}^{-1})_{mn} \epsilon_{\tilde{v}n}, \quad v, \tilde{v} = 1, \ldots, V - 1 \tag{6.3}$$

is the determinant which comes from the integration over momenta, $Q(p, \alpha, \Theta)$ and $\tilde{Q}(p, \alpha, \Theta)$ are real-valued functions.[14]

[13] The lines of $G - \pi_n$ are: $\pi_1, \ldots, \pi_{n-1}, \pi_{n+1}, \pi_{n+2}, \ldots, \pi_I$. Since the total number of lines in $G - \pi_n$ is $I - 1$, one should actually have a different notation in the argument of P in Eq. (5.14): $\hat{\beta}_1 = \beta_1, \ldots, \hat{\beta}_{n-1} = \beta_{n-1}$, $\hat{\beta}_n = \beta_{n+1}, \ldots, \hat{\beta}_{I-1} = \beta_I$.

[14] The graph-theoretic formula for $P(G, \theta)$ was given in ref.[4, 5]. Q and $\tilde{Q}$ can be expressed in terms of $P(G, \theta)$ [5].

Let us briefly outline the proof of the convergence theorem. In Eq. (6.1), only the convergence at the origin $\alpha = 0$ is questionable. Thus we consider the following part of the integrand in Eq. (6.1)

$$\frac{\exp[-Q(p,\alpha,\Theta)]}{\prod_{i=1}^{d/2} P(G,\theta_i)} \tag{6.4}$$

Let us perform the change of variables Eq. (5.5) in the above equation. According to theorem 1 to be proven below, the following relation holds around the origin $\beta = 0$

$$P(G,\theta) = \theta^{c(G)} \left(\prod_{i=1}^{I} \beta_i^{2(L(\gamma_i)-c_G(\gamma_i))} \right) [1 + \mathcal{O}(\beta^2)] \tag{6.5}$$

where $c(G) = 2g$ is twice the genus of the graph G.

In theorem 2 it will be shown that

$$Q(p,\alpha,\Theta) = \left(\prod_{i=1}^{I} \beta_i^{-2j(\gamma_i)} \right) [f_G(\pi,p,\Theta) + \mathcal{O}(\beta^2)] \tag{6.6}$$

where f_G is a function which depends on external momenta p of graph G, Θ and the permutation π from Eq. (5.5). The function f_G is non-negative $f_G \geq 0$. For the non-exceptional external momenta p it is strictly positive: $f_G > 0$.

Combining Eq. (6.4), Eq. (6.5), Eq. (6.6) and Eq. (5.6), it is easy to see that the integrand of Eq. (6.1) is majorized by the factor

$$\exp\left(-f_G(\pi,p,\Theta) \prod_{i=1}^{I} \beta_i^{-2j(\gamma_i)} \right) \prod_{l=1}^{I} \beta_l^{-[\omega(\gamma_l)-c_G(\gamma_l)d]-1} \tag{6.7}$$

The convergence theorem follows from Eq. (6.7).

6.2 Lemmas and theorems

In this section we prove four lemmas and three theorems which are inter-related as follows:

$$\begin{tikzcd}
 & & \text{Lemma 6.3} \arrow[d] & \\
\text{Lemma 6.1} \arrow[r] & \text{Lemma 6.2} \arrow[r] & \text{Theorem 1} \arrow[r] \arrow[d] & \text{Theorem 2} \arrow[dl] \\
 & \text{Lemma 6.4} \arrow[r] & \text{Theorem 3} &
\end{tikzcd}$$

In lemma 6.1 we prove a non-commutative analog of relation Eq. (5.10). For this purpose we need the following representation of $P(G,\theta)$ from Eq. (6.3). Using the Cauchy-Binet theorem and the Jacobi ratio theorem, $P(G,\theta)$ can be expressed as a double sum

over all possible trees $\mathcal{T}(G)$ of the graph G as follows [15]

$$P(G,\theta) = \sum_{T,T' \in \mathcal{T}(G)} (-)^{T+T'} \det \epsilon[T] \ \det \epsilon[T'] \ \det \mathcal{A}[T^*|T'^*] \tag{6.8}$$

where $\epsilon[T]$ denotes the minor of the incidence matrix ϵ corresponding to the lines in the tree T, while T^* denotes the complement of the tree T, i.e. $T^* = G \setminus T$. The quantity $(-)^{T+T'}$ is defined as follows. If

$$T = \{i_1, i_1, ..., i_{V-1}\}, \qquad T' = \{j_1, j_1, ..., j_{V-1}\}$$

then[16]

$$(-)^{T+T'} \equiv (-)^{\sum_{k=1}^{V-1} i_k + j_k} \tag{6.9}$$

Lemma 6.1 *If G is an L loop graph and l is one of its lines, then*

$$P(G,\theta) = \begin{cases} P(G-l,\theta) & \text{if } L(G-l) = L(G) \\ \alpha_l \, P(G-l,\theta) + X(G|l) & \text{if } L(G-l) = L(G) - 1 \end{cases} \tag{6.10}$$

where $X(G,\theta|l) = P(G,\theta)|_{\alpha_l=0}$.

Proof.

If $L(G-l) = L(G)$, then the line l belongs to all trees of the graph G and thus $P(G,\theta)$ does not depend on α_l. It implies that $P(G,\theta) = P(G-l,\theta)$.

Consider now the case when $L(G-l) = L(G) - 1$. Let us take the derivative of both sides of Eq. (6.8) with respect to α_l. It is clear that some terms will not give any contribution since $\frac{\partial}{\partial \alpha_l} \det \mathcal{A}[T^*|T'^*] = 0$ if $l \in T$ or $l \in T'$. Thus, we have

$$\frac{\partial}{\partial \alpha_l} P(G,\theta) = \sum_{\substack{T,T' \in \mathcal{T}(G) \\ l \notin T \cup T'}} (-)^{T+T'} \det \epsilon[T] \det \epsilon[T'] \frac{\partial}{\partial \alpha_l} \det \mathcal{A}[T^*|T'^*] \tag{6.11}$$

By Laplace expanding $\det \mathcal{A}[T^*|T'^*]$ about the row containing α_l we see that the only piece that survives is the minor corresponding to α_l, *i.e.*

$$\frac{\partial}{\partial \alpha_l} \det \mathcal{A}[T^*|T'^*] = \det \hat{\mathcal{A}}[T^*|T'^*] \tag{6.12}$$

[15]For an analogous representation in the commutative case see ref.[16]. Note that this representation assumes that G has no tadpoles. The graphs containing tadpoles can be treated as follows. If a line l begins and ends on the same vertex v, replace it by two lines l and l' forming a loop. Call the resulting graph G'. The original graph G can be obtained from G' by shrinking line l'. Now in the sum over trees Eq. (6.8) the lines l and l' will enter symmetrically. In other words, the associated Schwinger parameters will always come in the combination $(\alpha_l + \alpha_{l'})$. This means that the relations which follow from Eq. (6.8) are also true for the graphs containing tadpoles. In particular, lemma 6.1 holds for the graphs with tadpoles as well as those without.

[16]It is assumed that $i_1 < i_2 < \cdots < i_{V-1}$ and $j_1 < \cdots < j_{V-1}$.

where $\hat{\mathcal{A}}$ is obtained from $\mathcal{A}$ by removing the row and the column containing α_l. Using the above relation the r.h.s. of Eq. (6.11) becomes

$$\begin{aligned}
&\sum_{\substack{T,T'\in\mathcal{T}(G)\\ l\notin T\cup T'}} (-)^{T+T'} \det\epsilon[T]\det\epsilon[T']\det\hat{\mathcal{A}}[T^*|T'^*] \\
&\quad= \sum_{T,T'\in\mathcal{T}(G-l)} (-)^{T+T'} \det\epsilon[T]\det\epsilon[T']\det\mathcal{A}(G-l)[T^*|T'^*] \\
&\quad= P(G-l,\theta) \qquad (6.13)
\end{aligned}$$

where for the first equality we used the obvious fact that if a line does not belong to T and T^* of a graph G, then it does not belong to G. **q.e.d.**

For later purposes, let us introduce the following notation:

$$X(G,\theta|M) = P(G,\theta)|_{\alpha_l=0,\ \forall l\in M} \qquad (6.14)$$

where M is a set of lines of the graph G. This notation will be useful in Section 7.

Consider now the following question. Assume that we rescale all of the Schwinger parameters except the one associated to the line l by the same factor ρ. What is the scaling of $P(G,\theta)$ as $\rho\to 0$? Before answering this question, let us consider a simpler problem of rescaling all of the Schwinger parameters by ρ.

It is easy to see from Eq. (6.8) that the smallest number of α's in the monomials forming $P(G,\theta)$ equals the number of loops $L(G)$ minus the rank $r(G)$ of the intersection matrix I_{mn}. Thus, $P(G,\theta)$ is a polynomial of degree $r(G)$ in θ:

$$P(G,\theta) = \sum_{n=0}^{r(G)/2} \theta^{2n} P_{2n}(G) \qquad (6.15)$$

By the dimensional argument we find from Eq. (6.15) the following scaling

$$P(\rho G,\theta) = \rho^{L(G)-r(G)}\ \left[P_r(G) + \mathcal{O}(\rho)\right] \qquad (6.16)$$

where the notation ρG stands for the rescaling of all Schwinger parameters α by ρ. Note that the rank $r(G)$ of the intersection matrix of a graph G is equal to twice the genus of the graph G.[4, 5]. From the definition of $c(G)$ it follows that

$$r(G) = c_G(G) \qquad (6.17)$$

We are interested in the scaling of $P_r(G)$ only, because it gives the leading singular term in the integrand of Feynman integral.

Lemma 6.2 *Let l be a line of a graph G. In the expression for $P_r(G)$ let us rescale all Schwinger parameters except the one associated to the line l: $\alpha_j \to \rho\alpha_j$ for all $j\neq l$. Then the following relations hold:*

(1) If $L(G-l) = L(G)$, then

$$P_r(\alpha_l, \rho(G-l)) = \rho^{L(G)-r(G)} P_r(G) \tag{6.18}$$

(2) If $L(G-l) \neq L(G)$ and $r(G-l) < r(G)$, then Eq. (6.18) holds.

(3) If $L(G-l) \neq L(G)$ and $r(G-l) = r(G)$, then

$$P_r(\alpha_l, \rho(G-l)) = \rho^{L(G-l)-r(G)} \left[\alpha_l P_r(G-l) + \mathcal{O}(\rho)\right] \tag{6.19}$$

Proof.

(1) $L(G-l) = L(G)$

From lemma 6.1 we have $P(G,\theta) = P(G-l,\theta)$. Thus Eq. (6.18) holds.

(2)$L(G-l) \neq L(G)$ and $r(G-l) < r(G)$

From the lemma 6.1 we have in this case

$$P(G,\theta) = \alpha_l P(G-l,\theta) + X(G|l)$$

By taking $r(G)$-th derivative w.r.t. θ of both sides of this equation and using the fact that $P_{r(G)}(G-l) = 0$ (which follows from the condition $r(G-l) < r(G)$), we find

$$P_r(G) = \frac{1}{r!}\partial_\theta^r X(G|l) \tag{6.20}$$

Thus $P_r(G)$ is independent of α_l. Thus Eq. (6.18) holds in this case.

(3) $L(G-l) \neq L(G)$ and $r(G-l) = r(G)$

From the lemma 6.1 we have in this case

$$P(G,\theta) = \alpha_l P(G-l,\theta) + X(G|l)$$

By taking $r(G)$-th derivative w.r.t. θ of both sides of this equation we get

$$P_r(G) = \alpha_l\, P_r(G-l) + \frac{1}{r!}\partial_\theta^r X(G|l) \tag{6.21}$$

Rescaling all α except α_l yields

$$P_r(\alpha_l, \rho(G-l)) = \rho^{L(G-l)-r(G-l)} \left[\alpha_l P_r(G-l) + \mathcal{O}(\rho)\right] \tag{6.22}$$

since, according to lemma 6.1, $X(G|l)$ is independent of α_l. **q.e.d.**

Lemma 6.3 *Let G be a NQFT Feynman graph with the external momenta set to zero. Consider a set of lines $l_1, l_2, \ldots, l_k \in G$. Let us define $G_i,\ i = 1,\ldots,k$ as*

$$G_k \equiv G, \qquad G_{i-1} = G_i - l_i, \quad i = 1,\ldots,k \tag{6.23}$$

Let K_1 and K_2 be the sets of lines of G defined by Eq. (3.1) and Eq. (3.2). If $l_1, l_2, \ldots, l_k \in K_1(G) \cup K_2(G)$, then the following relations hold

$$L(G_i) - c_G(G_i) = L(G) - c_G(G), \qquad i = 0, \ldots, k \tag{6.24}$$

Remark. At this point one might find it useful to check the relations Eq. (6.24) for a particular graph. Consider the graph G in figure 6. We have $K_2(G) = \{3, 4, 5, 6, 7, 8\}$. Let us take $l_1 = 8$, $l_2 = 7$ and $l_3 = 6$. Then

$$G_3 = G, \quad G_2 = \{1, 2, 3, 4, 5, 7, 8\},$$

$$G_1 = \{1, 2, 3, 4, 5, 8\}, \quad G_0 = \{1, 2, 3, 4, 5\}$$

Since

$$L(G_3) = 5, \quad L(G_2) = 4, \quad L(G_1) = 3, \quad L(G_0) = 2$$

and

$$c_G(G_3) = 4, \;\; c_G(G_2) = 3, \;\; c_G(G_1) = 2, \;\; c_G(G_0) = 1$$

the relations Eq. (6.24) are satisfied.

Proof.

The proof relies on two auxiliary statements:

(1) Let $\mathcal{L} \subset G$ be any loop which contains a line $l \in K_2(G)$. Then the associated cycle $\mathcal{C}(\mathcal{L}) \in H_1(\Sigma(G))$ is non-trivial.[17]

Suppose that $\mathcal{C}(\mathcal{L})$ is trivial. The condition $l \in K_2(G)$ implies that there exists a loop $\mathcal{L}' \ni l$ containing l with the property that the associated cycle $\mathcal{C}(\mathcal{L}')$ is non-trivial and $\mathcal{C}(\mathcal{L}'') \neq \mathcal{C}(\mathcal{L}')$ for any loop $\mathcal{L}'' \not\ni l$. Using the rule of addition of cycles it is easy to see that $\mathcal{C}(\mathcal{L} \cup \mathcal{L}' - l) = \mathcal{C}(\mathcal{L}')$. This is a contradiction, since $\mathcal{L} \cup \mathcal{L}' - l$ does not contain the line l. Thus $\mathcal{C}(\mathcal{L})$ is non-trivial.

(2) If $l_1, l_2 \in K_2(G)$ and $c_G(G - l_1 - l_2) = c_G(G - l_2)$, then $L(G - l_1 - l_2) = L(G - l_2)$.

Suppose that $L(G - l_1 - l_2) \neq L(G - l_2)$, i.e. l_1 belongs to a loop in $G - l_2$. Let $\mathcal{L}$ be such a loop. The condition $l_1 \in K_2(G)$ and the statement (1) implies that $\mathcal{C}(\mathcal{L})$ is non-trivial. The condition $c_G(G - l_1 - l_2) = c_G(G - l_2)$ implies that there is a loop $\mathcal{L}' \subset G - l_2$ with the properties: (a) $l_1 \notin \mathcal{L}'$, (b) $\mathcal{C}(\mathcal{L}')$ is non-trivial and $\mathcal{C}(\mathcal{L}') = \mathcal{C}(\mathcal{L}) + \Delta\mathcal{C}$, where $\Delta\mathcal{C} \in H_1(\Sigma(G))$ is associated to a loop in $G - l_1 - l_2$.

The condition $c_G(G - l_2) < c_G(G)$ implies that for any loop $\mathcal{L}'' \subset G$, $\mathcal{L}'' \ni l_2$, the following statement holds: $\mathcal{C}(\mathcal{L}'')$ is not a linear combination of the homology cycles formed by the loops of $G - l_2$. In other words, $\mathcal{C}(\mathcal{L}'')$ is independent of the homology cycles associated to loops of $G - l_2$.

[17]For the definition of $\mathcal{C}(\mathcal{L})$ see the remark following definition 1.

Let us compare the numbers of independent homology cycles $c_G(G)$ and $c_G(G-l_1)$. For this purpose let us choose an arbitrary loop $\mathcal{L}'' \subset G$, $\mathcal{L}'' \ni l_2$. In G there are $\mathcal{C}(\mathcal{L})$, $\mathcal{C}(\mathcal{L}'')$ and some other cycles on which $\mathcal{C}(\mathcal{L})$ and $\mathcal{C}(\mathcal{L}'')$ are independent of. Which cycles do we have in $G - l_1$? There are two cases to consider: (a) $l_1 \notin \mathcal{L}''$ and (b) $l_1 \in \mathcal{L}''$. In the case (a) we have $\mathcal{C}(\mathcal{L}') = \mathcal{C}(\mathcal{L}) + \Delta\mathcal{C}$ and $\mathcal{C}(\mathcal{L}'')$. In the case (b) we have $\mathcal{C}(\mathcal{L}') = \mathcal{C}(\mathcal{L}) + \Delta\mathcal{C}$ and $\mathcal{C}(\mathcal{L}) + \mathcal{C}(\mathcal{L}'')$. Thus in both cases the number of independent cycles of $G - l_1$, $c_G(G-l_1)$, is equal to $c_G(G)$. This is a contradiction to the assumption $l_1 \in K_2(G)$. Thus $L(G - l_1 - l_2) = L(G - l_2)$.

By assumption we have $l_1, \ldots, l_k \in K_1(G_k) \cup K_2(G_k)$. Let us remove l_k from G_k and see what happens to a line $l_i \in K_2(G_k)$ $(i < k)$. There are only three possibilities:

(a) $l_i \in K_1(G_{k-1})$

(b) $l_i \in K_2(G_{k-1})$

(c) $l_i \notin K_1(G_{k-1})$ and $l_i \notin K_2(G_{k-1})$

Let B_{k-1} be the set of lines $l_i \in K_2(G_k)$ $(i < k)$ with the property (c). Thus we have

$$l_1, \ldots, l_{k-1} \in K_1(G_{k-1}) \cup K_2(G_{k-1}) \cup B_{k-1} \tag{6.25}$$

Similarly, the set B_{i-1} $(1 \leq i < k)$ is defined as follows. Let us remove l_i from G_i and see what happens to a line $l_j \in K_2(G_i) \cup B_i$ $(j < i)$. There are only three possibilities:

(a') $l_j \in K_1(G_{i-1})$

(b') $l_j \in K_2(G_{i-1})$

(c') $l_j \notin K_1(G_{i-1})$ and $l_j \notin K_2(G_{i-1})$.

Let B_{i-1} be the set of lines $l_j \in K_2(G_i) \cup B_i$ $(j < i)$ with the property (c'). Thus we have

$$l_1, \ldots, l_{i-1} \in K_1(G_{i-1}) \cup K_2(G_{i-1}) \cup B_{i-1} \tag{6.26}$$

From the above definition of B_i $(0 \leq i < k-1)$ it follows that $B_i \subset K_2(G)$.

The lemma follows from the following two statements:

(3) If $L(G_{i-1}) = L(G_i)$, then $c_G(G_{i-1}) = c_G(G_i)$.

This statement is evident. The condition $L(G_{i-1}) = L(G_i)$ (or, equivalently, $l_i \in K_1(G_i)$) implies that there is no loop in G_i to which l_i belongs to. Therefore, no cycle can be lost by removing l_i from G_i. Thus, $c_G(G_{i-1}) = c_G(G_i)$.

(4) If $L(G_{i-1}) = L(G_i) - 1$, then $c_G(G_{i-1}) = c_G(G_i) - 1$.

The condition $L(G_{i-1}) = L(G_i) - 1$ implies that $l_i \in K_2(G_i)$ or $l_i \in B_i$. If $l_i \in K_2(G_i)$ then, by the definition of the set $K_2(G_i)$, we have $c_G(G_{i-1}) < c_G(G_i)$. But by disconnecting a single loop the cycle number can decrease by at most one. Thus we have $c_G(G_{i-1}) = c_G(G_i) - 1$. If $l_i \in B_i$ then according to statement (2) by removing l_i the independent cycle associated to loops in G_i containing l_i will be lost. This implies again that $c_G(G_{i-1}) = c_G(G_i) - 1$. **q.e.d.**

Theorem 1 *Let G be a general NQFT Feynman graph. Consider the change of variables Eq. (5.5) specified by a permutation π. Let Π be the set of all permutations of $\{1, 2, \ldots, I\}$. Then for any $\pi \in \Pi$ the following relation holds*

$$P_{2g}(G) = \left(\prod_{i=1}^{I} \beta_i^{2(L(\gamma_i) - c_G(\gamma_i))} \right) [1 + \mathcal{O}(\beta^2)] \tag{6.27}$$

where $g = r(G)/2$.

Remark. At this point one might find it useful to check Eq. (6.27) for some particular graphs, say for the figure 9(a). The homogeneous polynomial P_4 from Eq. (2.5) for the latter graph is

$$P_4 = (\alpha_1+\alpha_2+\alpha_3+\alpha_4)(\alpha_5+\alpha_7)(\alpha_8+\alpha_9)+(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)(\alpha_5+\alpha_7+\alpha_8+\alpha_9) \tag{6.28}$$

Consider the subgraph formed by the lines $1, 2, 3, 4$. Let us rescale the corresponding Schwinger parameters by t:

$$\alpha_1, \alpha_2, \alpha_3, \alpha_4 \to t\alpha_1, t\alpha_2, t\alpha_3, t\alpha_4$$

It is easy to see that $P_4 \sim t = t^{2-1}$ for $t \sim 0$. Similarly, for any subgraph γ we have $P_4 \sim t^{L(\gamma)-c(\gamma)}$.

Proof.

We will prove the relation Eq. (6.27) by induction. It is not difficult to check that Eq. (6.27) holds for all one and two loop diagrams.

Let us assume that Eq. (6.27) holds for all diagrams with number of loops less or equal to L and prove the relation Eq. (6.27) for diagrams with $L+1$ loops. Let G be an arbitrary $L+1$ graph. We will prove that Eq. (6.27) holds for G.

Let I be the total number of lines of the graph G. Consider the change of variables Eq. (5.5) specified by a permutation π.

If all the lines $1, \ldots, I$ of the graph G are from the set $K_1(G) \cup K_2(G)$, then $L(\gamma_i) = c(\gamma_i)$ for all $i \in \{1, \ldots, I\}$, since any loop of G wraps a non-trivial cycle. Using the dimensional argument, we see from Eq. (6.15) that $P_{2g}(G)$ does not depend on α. More precisely,

$$P_{2g}(G) = \det\, I \tag{6.29}$$

where I is a nondegenerate intersection matrix of size $2g \times 2g$. Changing the basis to the canonical one: $\{a_1, b_1, \ldots, a_g, b_g\}$, it is easy to see that $P_{2g}(G) = 1$.

Thus let us consider the case when there is at least one line which does not belong to the set $K_1(G) \cup K_2(G)$. Let π_n be the first line in the sequence $\pi_I, \pi_{I-1}, \ldots, \pi_{n+1}, \pi_n, \ldots$, which does not belong to $K_1(G) \cup K_2(G)$, i.e.

$$\pi_n \notin K_1(G) \cup K_2(G), \qquad \pi_i \in K_1(G) \cup K_2(G), \quad n+1 \leq i \leq I \tag{6.30}$$

From the lemma 6.2 and Eq. (6.30) it follows that $P_{2g}(G)$ does not depend on $\alpha_{\pi_{n+1}}, \ldots, \alpha_{\pi_I}$. Thus $(\beta_{n+1}\beta_{n+2}\cdots\beta_I)$ trivially factors out of $P_{2g}(G)$:

$$P_{2g}(G) = (\beta_{n+1}\beta_{n+2}\cdots\beta_I)^{2(L(G)-c_G(G))} P_{2g}(G|\beta_{n+1}=1,\ldots,\beta_I=1) \tag{6.31}$$

Since $P_{2g}(G)$ is a homogeneous polynomial of degree $L(G)-c_G(G)$ in $\alpha_{\pi_1},\ldots,\alpha_{\pi_n}$, we have

$$P_{2g}(G|\beta_{n+1}=1,\ldots,\beta_I=1) = \beta_n^{2(L(G)-c_G(G))} P_{2g}(G|\beta_n=1,\ldots,\beta_I=1) \tag{6.32}$$

Using lemma 6.3 one can show that

$$(\beta_{n+1}\beta_{n+2}\cdots\beta_I)^{2(L(G)-c_G(G))} = \prod_{i=n+1}^{I} \beta_i^{2(L(\gamma_i)-c_G(\gamma_i))} \tag{6.33}$$

From the case 3 of lemma 6.2 (with $l=\pi_n$ and $\rho=\beta_{n-1}^2$) we find

$$\begin{aligned} &P_{2g}(G|\beta_n=1,\ldots,\beta_I=1) \\ &\quad = \beta_{n-1}^{2(L(G-\pi_n)-c_{G-\pi_n}(G-\pi_n))}[P_{2g}(G-\pi_n|\beta_{n-1}=1,\ldots,\beta_I=1)+\mathcal{O}(\beta_{n-1}^2)] \end{aligned} \tag{6.34}$$

The second term in the above equation is subleading in β_{n-1}^2. Thus we consider the first term. It can be rewritten as

$$\beta_{n-1}^{2(L(G-\pi_n)-c_{G-\pi_n}(G-\pi_n))} P_{2g}(G-\pi_n|\beta_{n-1}=1,\ldots,\beta_I=1) = P_{2g}(G-\pi_n|\beta_n=1,\ldots,\beta_I=1) \tag{6.35}$$

since $P_{2g}(G-\pi_n)$ is a homogeneous polynomial of degree $L(G-\pi_n)-c_{G-\pi_n}(G-\pi_n)$ in α. Since $G-\pi_n$ is an L loop graph, Eq. (6.27) holds for it. If a line $l\in G$ is from $K_1(G)\cup K_2(G)$, then $l\in K_1(G-\pi_n)\cup K_2(G-\pi_n)$. Thus it is easy to see that $P_{2g}(G-\pi_n)$ does not depend on $\alpha_{\pi_{n+1}},\ldots,\alpha_{\pi_I}$. We thus have

$$P_{2g}(G-\pi_n|\beta_{n+1}=1,\ldots,\beta_I=1) = \left(\prod_{i=1}^{n-1}\beta_i^{2(L_{G-\pi_n}(\gamma_i)-c_{G-\pi_n}(\gamma_i))}\right)[1+\mathcal{O}(\beta)] \tag{6.36}$$

where $L_{G-\pi_n}(\gamma)$ and $c_{G-\pi_n}(\gamma)$ are the number of loops and the cycle numbers of graph γ with respect to the graph $G-\pi_n$. It is clear that for any subgraph $\gamma\subset G$ which does not contain line π_n, we have

$$L_G(\gamma) = L_{G-\pi_n}(\gamma) \tag{6.37}$$

From the assumption $g(G-\pi_n)=g(G)$ it follows that for any subgraph $\gamma\subset G$ which does not include line π_n we have

$$c_{G-\pi_n}(\gamma) = c_G(\gamma) \tag{6.38}$$

since by removing the line π_n we do not disconnect any cycle of $\Sigma_g(G)$. **q.e.d.**

Note that theorem 1 together with Eq. (6.16) implies that Eq. (6.5) holds.

Theorem 2 *Let G be a 1PI graph. Consider the change of variables Eq. (5.5) specified by a permutation π. Let Π be the set of all permutations of $\{1,2,\ldots,I\}$. Then for any $\pi \in \Pi$ the following relation holds*

$$Q(p,\alpha,\Theta) = \left(\prod_{i=1}^{I} \beta_i^{-2j(\gamma_i)}\right) [f_G(\pi,p,\Theta) + \mathcal{O}(\beta^2)] \tag{6.39}$$

where f_G is a non-negative function

$$f_G(\pi,p,\Theta) \geq 0. \tag{6.40}$$

Proof.

Let G_{ij} be the graph obtained from G by connecting external lines i and j (see figure 10). As shown in appendix A (see also ref.[5]) Q is given by

$$Q(p,\alpha,\Theta) = -\sum_{i\neq j} p_i^{\mu} \left(\frac{P(G_{ij},\Theta|\alpha_{ij}=0)}{P(G,\Theta)}\right)^{\mu\nu} p_j^{\nu} \tag{6.41}$$

and it is non-negative

$$Q(p,\alpha,\Theta) \geq 0 \tag{6.42}$$

For a given permutation π, let γ_{k_0} in Eq. (5.4) be the smallest subgraph such that $j(\gamma_{k_0}) = 1$. In other words,

$$\begin{aligned} j(\gamma_l) &= 0, \quad 1 \leq l \leq k_0 - 1 \\ j(\gamma_l) &= 1, \quad k_0 \leq l \leq I \end{aligned} \tag{6.43}$$

Let i,j be two external lines of G such that

$$c_{G_{ij}}(\gamma_{k_0}) = c_G(\gamma_{k_0}) + 1 \tag{6.44}$$

The existence of at least a pair of such lines follows from the fact that $j(\gamma_{k_0}) = 1$. From theorem 1 we have[18]

$$P(G_{ij},\theta|\alpha_{ij}=0) = \theta^{c(G)+2} \left(\prod_{l=1}^{I} \beta_l^{2(L_{G_{ij}}(\gamma_l) - c_{G_{ij}}(\gamma_l))}\right) [1 + \mathcal{O}(\beta^2)] \tag{6.45}$$

where subscript G_{ij} in $L_{G_{ij}}$ and $c_{G_{ij}}$ means that the corresponding quantities are defined with respect to the graph G_{ij}. Combining the above equation with Eq. (6.5) and using the relations

$$L_{G_{ij}}(\gamma_l) = L_G(\gamma_l), \quad 1 \leq l \leq I \tag{6.46}$$

[18]For the graph G_{ij} one should restrict to the subset of permutations $\{\pi|\pi_{I+1} = l_{ij}\}$, where l_{ij} is the new line formed by the joining of external lines i and j. Note that, by assumption Eq. (6.44), we have $g(G_{ij}) > g(G)$. Thus, according to lemma 6.2, $P_{2g(G_{ij})}(G_{ij})$ is independent of α_{ij}. If $j(G) = 0$, then $g(G_{ij}) = g(G)$ for any $i \neq j$ and $P_{2g(G_{ij})}(G_{ij})$ depends on α_{ij}. It is clear that if $g(G_{ij}) = g(G)$, then for any subgraph $\gamma \subset G$ we have $c_{G_{ij}}(\gamma) = c_G(\gamma)$. Thus, if $j(G) = 0$, then for any i,j the leading term in $P(G_{ij},\theta|\alpha_{ij}=0)$ is subleading compared to the leading term in $P(G,\theta)$.

and

$$c_{G_{ij}}(\gamma_l) = c_G(\gamma_l), \quad 1 \le l \le k_0 - 1$$
$$c_{G_{ij}}(\gamma_l) = c_G(\gamma_l) + 1, \quad k_0 \le l \le I \tag{6.47}$$

we find

$$\frac{P(G_{ij}, \theta | \alpha_{ij} = 0)}{P(G, \theta)} = \theta^2 \left(\prod_{l=1}^{I} \beta_l^{-2j(\gamma_l)} \right) [1 + \mathcal{O}(\beta^2)] \tag{6.48}$$

The r.h.s. of the above equation does not depend on the particular pair of lines i, j that we chose as long as $c_{G_{ij}}(\gamma_{k_0}) > c(\gamma_{k_0})$. It means that the leading contribution to Q of any pair of lines i, j which satisfy Eq. (6.44) is

$$-p_i^\mu (\Theta^2)^{\mu\nu} p_j^\nu \left(\prod_{l=1}^{I} \beta_l^{-2j(\gamma_l)} \right)$$

Let Y be the set of all ordered pairs of external lines of graph G which satisfy condition Eq. (6.44). It is easy to see that Eq. (6.39) holds, with the function f_G given by

$$f_G(\pi, p, \Theta) = - \sum_{(i,j) \in Y} p_i^\mu (\Theta^2)^{\mu\nu} p_j^\nu \tag{6.49}$$

The relation Eq. (6.40) follows from Eq. (6.42) and the following statement. Let $F(\beta) \ge 0$ be a non-negative function for $\beta \ge 0$. Then the leading term in the expansion of $F(\beta)$ around $\beta = 0$ is non-negative. **q.e.d**

Lemma 6.4 *For any 1PI graph G, there exists a unique decomposition*

$$\mathcal{E}(G) = \bigcup_{s=1}^{n(\gamma)} \mathcal{E}_s(\gamma) \tag{6.50}$$

for some $n(\gamma) \ge 1$.

Proof.

If $j(\gamma) = 0$, then the lemma holds trivially with $n(\gamma) = 1$. Thus consider the case $j(\gamma) = 1$. From the definition 2 of index j it follows that there exists a pair of external lines i, k such that $c_{G_{ik}}(\gamma) > c_G(\gamma)$. Now, let us look for all external lines $l \ne i$ with the property $c_{G_{il}}(\gamma) = c_G(\gamma)$. Denote by $\mathcal{E}_1$ the latter set of lines together with the line i.

Let us show that for any line l from the set $\mathcal{E}_1$ we have $c_{G_{kl}}(\gamma) > c_G(\gamma)$. From the fact that $c_{G_{ik}}(\gamma) > c_G(\gamma)$ one can infer that there exists a loop $\mathcal{L}$ around the new handle which was created after joining external lines i and k as in figure 44. From the fact that $c_{G_{il}}(\gamma) = c_G(\gamma)$ for all $l \in \mathcal{E}_1$ we can infer that $\mathcal{L}$ encloses all the lines from the set $\mathcal{E}_1$. Thus we have $c_{G_{kl}}(\gamma) > c_G(\gamma)$ for all $l \in \mathcal{E}_1$. **q.e.d.**

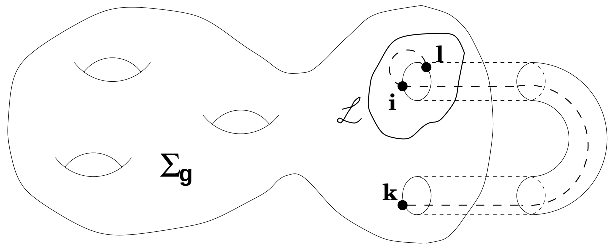


Figure 44: Graph G_{ik}.

Theorem 3 *Let I_G be the Feynman integral associated with a 1PI graph G in a massive non-derivative scalar quantum field theory over non-commutative $\mathbb{R}^d$. There are three cases:*

(I) If $j(G) = 0$, then I_G is convergent if and only if $\omega(\gamma) - c_G(\gamma)d < 0$ for all $\gamma \subseteq G$.

(II) If $j(G) = 1$ and the external momenta are non-exceptional, then I_G is convergent if for any subgraph $\gamma \subseteq G$ at least one of the following conditions is satisfied:(1) $\omega(\gamma) - c_G(\gamma)d < 0$, (2) $j(\gamma) = 1$.

(III) If $j(G) = 1$ and the external momenta are exceptional, then I_G is convergent if $\omega(\gamma) - c_G(\gamma)d < 0$ for all $\gamma \subseteq G$.

Proof.

Case I. As explained earlier, Eq. (6.7) follows from theorem 1 and theorem 2. Since $j(G) = 0$, we have $j(\gamma) = 0$ for all subgraphs $\gamma \subset G$. Thus the exponential factor in Eq. (6.7) is independent of β. It is easy to see from the form of the prefactor in Eq. (6.7) that the theorem holds in this case.

Case II. In this case the exponential factor in Eq. (6.7) depends on β and it is capable of suppressing arbitrary powers of divergence coming from the prefactor in Eq. (6.7). Thus if $f_G(\pi, p, \Theta) > 0$ for all π, then I_G converges. It remains to show that $f_G(\pi, p, \Theta) = 0$ if and only if the external momenta p are exceptional.

f_G is given by Eq. (6.49). According to lemma 6.4, for any subgraph $\gamma \subseteq G$ there exists a unique decomposition Eq. (6.50) of the set of external lines of G.

Let us show that for the subgraph γ_{k_0} from theorem 2, the decomposition Eq. (6.50) has $n(\gamma_{k_0}) = 2$. By assumption (see theorem 2), for a given permutation π, γ_{k_0} is the smallest subgraph such that $j(\gamma_{k_0}) = 1$. Thus $n(\gamma_{k_0}) > 1$. Consider the subgraph γ_{k_0-1} which is obtained from γ_{k_0} by deleting line π_{k_0}. By assumption we have $j(\gamma_{k_0-1}) = 0$. Thus $n(\gamma_{k_0-1}) = 1$. By disconnecting a line, n may decrease by at most one.[19] We thus conclude that $n(\gamma_{k_0}) = 2$.

[19]This can be shown as follows. The lines from $\mathcal{E}_s(\gamma)$ are enlosed in the loops as in figure 44. By disconnecting a line the number of loops can decrease by at most one. Suppose that $\mathcal{L}_1$ and $\mathcal{L}_2$ enclose $\mathcal{E}_1(\gamma)$ and $\mathcal{E}_2(\gamma)$, respectively. Now remove a line l from γ. If $\mathcal{L}_1$ and $\mathcal{L}_2$ share the line l, then $\mathcal{E}_1(\gamma)$ and

Let us define momenta $\mathcal{P}_s(\gamma_{k_0}),\quad s=1,2$, as

$$\mathcal{P}_s(\gamma_{k_0}) = \sum_{i\in\mathcal{E}_s(\gamma_{k_0})} p_i \tag{6.51}$$

We thus have

$$f_G(\pi,p,\Theta) = 2\mathcal{P}_1^\mu(\gamma_k)(\Theta^2)^{\mu\nu}\mathcal{P}_1^\nu(\gamma_k) \tag{6.52}$$

Since Θ is assumed to be non-degenerate, Eq. (6.52) imples that $f_G=0$ if and only if $\mathcal{P}_1(\gamma_{k_0})=0$, i.e. the external momenta are exceptional.

Case III. If $\omega(\gamma)-c(\gamma)d<0$ for all $\gamma\subseteq G$, then it is easy to see from Eq. (6.7) that I_G will be convergent even if the exponent in Eq. (6.7) does not depend on β. **q.e.d.**

7 Convergence theorem for non-commutative theories–General case

In this section we prove the convergence theorem for the Feynman integrals in arbitrary quantum field theories over non-commutative $\mathbb{R}^d$ with the massive propagators. It is assumed that the reader is familiar with the constructions and proofs given sections 5 and 6. Thus we will omit the details of the proofs of some of the statements which are minor modifications of those of section 6.

7.1 Outline of the proof

Consider an arbitrary graph G in a general theory and let $\{q_l\}$ be the set of internal momenta which appear in the numerator of the Feynman integral. Let $\mathcal{I}_G$ be the integrand of the Feynman integral I_G in the α representation. Let us recall a trick introduced in ref.[5] which enables one to relate $\mathcal{I}_G$ to $\mathcal{I}_{G^+}$ for a graph G^+ in a non-derivative scalar theory.

The new graph G^+ is defined as follows. For each q^l, let v be the vertex which q leaves. Now include two new external momenta, a momentum r^l which leaves vertex v immediately counterclockwise from q and a momentum $-r^l$ which leaves q immediately clockwise from q. We will denote by l' and l'' the new external lines carrying momenta r^l and $-r^l$, respectively.[20] The only effect of these momenta is to multiply the integrand with an extra phase

$$e^{ir^l_\mu\Theta^{\mu\nu}q^l_\nu} \tag{7.1}$$

$\mathcal{E}_2(\gamma)$ will combine to form a set $\mathcal{E}_*(\gamma-l)$ from Eq. (6.50) for $\gamma-l$.

[20]From now on the letter r with an index l will be reserved for the momentum carried by the external lines l' and l'' attached to the line l with $\deg(l)>0$.

Thus, if the function of momenta in the numerator of the Feynman integral for G is some polynomial $P(q_\mu, p_\mu)$, we have

$$\mathcal{I}_G(\alpha,\Theta) = P(-i(\Theta^{-1})_{\mu\nu}\partial_{r^l_\nu}, p_\mu)\ \mathcal{I}_{G^+}(\alpha,\Theta)|_{r^l_\mu=0} \tag{7.2}$$

Let us denote by $\mathcal{E}'(G)$ the set of extra external lines added to G to make G^+, i.e.

$$\mathcal{E}'(G) = \mathcal{E}(G^+) \setminus \mathcal{E}(G) \tag{7.3}$$

The general form of the leading term in the integrand of the Feynman integral in β variables reads

$$\begin{aligned}&\left(\prod_{l=1}^{I}\beta_l^{2l-1}\right)\left(\prod_{l=1}^{I}\beta_l^{2(L(\gamma_l)-c_G(\gamma_l))}\right)^{-d/2}\\ &\times\left[\prod_{l\in G}\left(\frac{\partial}{\partial r_l}\right)^{\deg(l)}\right]\ \exp-\left[Q(p,\alpha,\Theta)+i\tilde{Q}(p,\alpha,\Theta)\right]\Big|_{r=0}\end{aligned} \tag{7.4}$$

In lemma 7.8 it will be shown that $\tilde{Q}$ is $\mathcal{O}(1)$ in the β expansion. Thus, as far as the singular terms are concerned, we can omit $\tilde{Q}(p,\alpha,\Theta)$ in the exponent in Eq. (7.4).

In lemma 7.1 it will be shown that if $l \in K_2(G)$, then the r_l-dependent terms in $Q(p,\alpha,\Theta)$ are nonsingular. Thus, as far as the singular terms are concerned, Eq. (7.4) can equivalently be written in the form

$$\begin{aligned}&\left(\prod_{l=1}^{I}\beta_l^{2l-1}\right)\left(\prod_{l=1}^{I}\beta_l^{2(L(\gamma_l)-c_G(\gamma_l))}\right)^{-d/2}\\ &\times\left[\prod_{l\in G}\left(\frac{\partial}{\partial r_l}\right)^{\deg(l)\ \mathrm{ind}_{K_2}(l)}\right]\ \exp\left(\sum_{i,j}p_i^\mu\left(\frac{P(G_{ij},\Theta|\alpha_{ij}=0)}{P(G,\Theta)}\right)^{\mu\nu}p_j^\nu\right)\Big|_{r=0}\end{aligned} \tag{7.5}$$

where ind_{K_2} is defined in Eq. (3.5).

In general there are three types of terms p_ip_j in the exponent in Eq. (7.5):

(a) $i,j \in \mathcal{E}(G)$

(b) $i \in \mathcal{E}'(G)$, $j \in \mathcal{E}(G)$,

(c) $i,j \in \mathcal{E}'(G)$,

Let us consider these three cases.

Type a. These terms are independent of the momenta r_l and thus the derivatives in Eq. (7.5) do not act on them. These terms are considered in theorem 2. From theorem 2 we have

$$p_i(\Theta^2)p_j\prod_{l=k_0}^{I}\beta_l^{-2} \tag{7.6}$$

Type b. Let γ_{k_0} be the subgraph considered in theorem 2. Consider the terms proportional to r_i in Eq. (7.5). In theorem 4 it will be shown that the leading term in the expansion of

$$\sum_j r_i Q_{ij} p_j \tag{7.7}$$

in β, after the summation over all $j \in \mathcal{E}(G)$ and using the momentum conservation for $p_j, \quad j \in \mathcal{E}(G)$, is of the form

$$\left(\prod_{l=k_1}^{I} \beta_l^{-2} \right) R_1(\beta_1, \beta_2, \ldots, \beta_{k_1-1}) \tag{7.8}$$

for some $k_1 \geq k_0$, where R_1 is a polynomial in variables $\beta_1, \beta_2, \ldots, \beta_{k_1-1}$.

Comparison of Eq. (7.8) and Eq. (7.6) shows that type **b** terms in the exponent in Eq. (7.5) are harmless because $k_1 \geq k_0$. Namely, the derivatives acting on the exponent in Eq. (7.5) will bring down type **b** terms and may, in principle, cause the divergence. But there will always be (as long as the external momenta of G are non-exceptional) the type **a** terms in the exponent which will suppress these potential divergences because $k_1 \geq k_0$. Thus, type **b** terms are harmless.

Type c. Let $\gamma_{k_{\bar{\imath}\bar{\jmath}}}$ be the smallest subgraph in Eq. (5.4) which contains both lines $\bar{\imath} \notin K_2$ and $\bar{\jmath} \notin K_2$ in loops. In lemma 7.9 it will be shown that the singular term in the exponent of Eq. (7.5) is of the form

$$\left(r_i(\Theta^2) r_j \prod_{l=k_2}^{I} \beta_l^{-2} \right) R_2(\beta_1, \beta_2, \ldots, \beta_{k_2-1}) \tag{7.9}$$

for some $k_2 \geq k_{\bar{\imath}\bar{\jmath}}$, where R_2 is a polynomial in variables $\beta_1, \beta_2, \ldots, \beta_{k_2-1}$.

From Eq. (7.9) it follows that the derivatives acting on the r^2 terms in the exponent in Eq. (7.5) cannot bring down the terms which are more singular than

$$\prod_{l=1}^{I} \frac{1}{\beta_l^{\#_l}} \tag{7.10}$$

where $\#_l$ is defined as the following sum over lines of the subgraph γ_l:

$$\#_l = \sum_{i \in \gamma_l} \deg(i) \operatorname{ind}_{K_2}(i) \tag{7.11}$$

Theorem 5 follows from the latter definition, the definition 5 for ω, Eq. (7.5) and the fact that type **b** terms are harmless.

7.2 Lemmas and theorems

In this section we prove nine lemmas and two theorems which are inter-related as follows:

$$
\begin{array}{ccccccccc}
 & & \text{Lemma 7.9} & & & \text{Lemma 7.2} & & & \\
 & & \downarrow & \nwarrow & & \swarrow & & & \\
\text{Lemma 7.1} & \rightarrow & \text{Theorem 5} & \leftarrow & \text{Theorem 4} & \leftarrow & \text{Lemma 7.4} & \leftarrow & \text{Lemma 7.3} \\
 & & \uparrow & \swarrow & & \nwarrow & & & \\
 & & \text{Lemma 7.8} & & & \text{Lemma 7.5} + \text{Lemma 7.6} + \text{Lemma 7.7} & & &
\end{array}
$$

Lemma 7.2 is a slight generalization of theorem 1 from section 6. Theorem 1 is used in lemma 7.9 and theorem 5.

Lemma 7.1 *Consider the expression*

$$Q = -\sum_{i,j} p_i^\mu \left(\frac{P(G_{ij}, \Theta | \alpha_{ij} = 0)}{P(G, \Theta)} \right)^{\mu\nu} p_j^\nu \tag{7.12}$$

Let $\bar{\imath}$ be an internal line of graph G with $\deg(\bar{\imath}) > 0$. There are three types of terms in Eq. (7.12): terms proportional to $r_{\bar{\imath}}^2$, terms proportional to $r_{\bar{\imath}}$ and terms independent of $r_{\bar{\imath}}$. If $\bar{\imath} \in K_2(G)$, then $r_{\bar{\imath}}$-dependent terms are of order $\mathcal{O}(1)$ in the β-expansion.

Proof.

Consider first $r_{\bar{\imath}}^2$ term. Let $i_1 \in \mathcal{E}'(G)$ and $i_2 \in \mathcal{E}'(G)$ be the external lines carrying momenta $r_{\bar{\imath}}$ and $-r_{\bar{\imath}}$, respectively. If $\bar{\imath} \in K_2(G)$, then there exists a cycle supported only by $\bar{\imath}$ (i.e. there exists an element $a_0 \in H_1(\Sigma_g(G))$ which is associated only to a loop containing the line $\bar{\imath}$). This implies that the joining of lines i_1 and i_2 will not introduce any extra cycle (i.e. the set of independent nontrivial cycles of the surface $\Sigma(G)$ associated with the graph G is the same as that of the surface $\Sigma(G_{i_1 i_2})$ associated with the graph $G_{i_1 i_2}$). Therefore, the coefficient of $r_{\bar{\imath}}^2$ in Eq. (7.12) is of order $\mathcal{O}(1)$ in the β-expansion (See footnote 18 for the discussion of this point).

Consider now term linear in $r_{\bar{\imath}}$. This term comes from the joining of i_1 or i_2 with some other external line $j \in \mathcal{E}(G^+)$. Since $\bar{\imath} \in K_2(G)$, $g(G_{i_1 j}) = g(G)$ if and only if $g(G_{i_2 j}) = g(G)$. If $g(G_{i_1 j}) = g(G)$, then the corresponding term in Eq. (7.12) is of order $\mathcal{O}(1)$ in β-expansion. Let us therefore consider the case $g(G_{i_1 j}) = g(G) + 1$. The assumption $\bar{\imath} \in K_2(G)$ implies that $g(G_{i_2 j}) = g(G) + 1$. The corresponding terms in Eq. (7.12) are proportional to

$$\frac{1}{P(G)} (P(G_{i_1 j}) - P(G_{i_2 j})) \tag{7.13}$$

$P(G)$ is given by Eq. (6.8). Let us analyze the difference

$$\left(\sum_{T,T'\in\mathcal{T}(G_{i_1j})}(-)^{T+T'}\det\epsilon[T]\ \det\epsilon[T']\det\mathcal{A}(G_{i_1j})[T^*|T'^*]\right)$$
$$-\left(\sum_{T,T'\in\mathcal{T}(G_{i_2j})}(-)^{T+T'}\det\epsilon[T]\ \det\epsilon[T']\det\mathcal{A}(G_{i_2j})[T^*|T'^*]\right) \tag{7.14}$$

Let l_* be the internal line of the graph G_{i_1j} (G_{i_2j}) formed by joining the external lines i_1 and j (i_2 and j) of G. For the simplicity of formulas, let us assume that $l_*=1$ and $\bar{\imath}=2$. The matrices $\mathcal{A}(G_{i_1j})$ and $\mathcal{A}(G_{i_2j})$ in Eq. (7.14) differ only in the matrix elements $\mathcal{A}_{12}=-\mathcal{A}_{21}$. It is easy to see that if the joining of i_1 with j creates the intersection of lines 1 and 2 in G_{i_1j}, , i.e. $I_{12}(G_{i_1j})=1$, then the lines 1 and 2 in G_{i_2j} do not intersect, i.e. $I_{12}(G_{i_2j})=0$. Similarly, we have: if $I_{12}(G_{i_2j})=1$, then $I_{12}(G_{i_1j})=0$. For simplicity we assume that $I_{12}(G_{i_1j})=1$.

Consider now Eq. (7.14). Since the matrices $\mathcal{A}(G_{i_1j})$ and $\mathcal{A}(G_{i_2j})$ differ only by the matrix elements $\mathcal{A}_{12}$ and $\mathcal{A}_{21}$, all but the terms proportional to I_{12} and I_{12}^2 cancel (because the lines i_1 and i_2 emerge from the same vertex there is a term by term correspondence between the two sums in Eq. (7.14)). Using the Laplace expansion along row 1 and column 1, it can be shown that

$$\begin{aligned}&\det\mathcal{A}[T^*|T'^*](G_{i_1j})-\det\mathcal{A}[T^*|T'^*](G_{i_2j})\\&\quad=\theta^2(I_{12}(G_{i_1j}))^2\det\mathcal{A}[T^*,\hat{1},\hat{2}|T'^*,\hat{1},\hat{2}](G_{i_1j})+\sum_{n\geq 3}(-)^n\theta^2 I_{12}(G_{i_1j})I_{1n}(G_{i_1j})\times\\&\qquad\times[\det\mathcal{A}[T^*,\hat{1},\hat{2}|T'^*,\hat{1},\hat{n}](G_{i_1j})+\det\mathcal{A}[T^*,\hat{1},\hat{n}|T'^*,\hat{1},\hat{2}](G_{i_1j})]\end{aligned} \tag{7.15}$$

where $\mathcal{A}[T^*,\hat{m}|T'^*,\hat{n}](G_{i_1j})$ denotes the matrix $\mathcal{A}[T^*|T'^*]$ with the rows and the columns corresponding to the lines m and n, respectively, removed. The following argument shows that $(I_{12})^2$ term in Eq. (7.15) is at most of the order $\theta^{2g(G)}$. The matrix $\mathcal{A}[T^*|T'^*](G_{i_1j})$ is a sum of two matrices (see Eq. (6.3)). Any rank $2g(G_{i_1j})$ minor of the intersection matrix $I(G_{i_1j})$ contains rows and columns corresponding to lines 1 and 2. The reason is the following. The matrix $I(G_{i_1j})$ with the rows 1,2 and columns 1,2 removed is the intersection matrix $I(G-\bar{\imath})$ for the graph $G-\bar{\imath}$. The rank of $I(G-\bar{\imath})$ is $2g(G)-2$ since $g(G-\bar{\imath})=g(G)-1$.[21]

[21] The rosette (see ref.[2, 4] for the definition) can be thought of as a set of curves on the Riemann surface $\Sigma_g(G)$, i.e. a set of maps $\phi_i: S^1\to\Sigma_g(G)$ with i running from 1 to the number of lines in the rosette. These maps can be decomposed in some orthonormal basis of the first homology group of $\Sigma_g(G)$: $\phi_i=\sum_{k=1}^{g}\tilde{a}_k\kappa_i^k+\sum_{k=1}^{g}\tilde{b}_k\rho_i^k$. The orthonormality is defined with respect to the scalar product given by the intersection form ($\langle\tilde{a}_i,\,\tilde{b}_j\rangle=\delta_{ij}$, $\langle\tilde{b}_i,\,\tilde{a}_j\rangle=-\delta_{ij}$, $\langle\tilde{a}_i,\,\tilde{a}_j\rangle=0$, $\langle\tilde{b}_i,\,\tilde{b}_j\rangle=0$). Such a basis has, of course, $2g$ independent elements. Because there are $2g$ independent elements in this vector space, the intersection matrix $I_{ij}=\langle\phi_i,\,\phi_j\rangle$ has rank $2g$.

Let us assume that in some basis of $H_1(\Sigma_g(G))$ there is a *unique* map ϕ_{i_0} which contains the basis element, say, $\tilde{a}_{i_0}$. If we construct the intersection matrix $I'_{ij}=\langle\phi_i,\,\phi_j\rangle$, $i,j\neq i_0$ for the rosette without

The matrix $\mathcal{A}[T^*, \hat{1}, \hat{2}|T'^*, \hat{1}, \hat{2}](G_{i_1 j})$ in Eq. (7.15) does not contain rows and columns corresponding to lines 1 and 2. Thus the determinant of this matrix is a polynomial of a degree not higher than $\theta^{2g(G)-2}$. Thus $(I_{12})^2$ term in Eq. (7.15) is at most of the order $\theta^{2g(G)}$. The same is true for the second term in Eq. (7.15).

Thus we see that the largest power of θ in the difference $P(G_{i_1 j}) - P(G_{i_2 j})$ is $\theta^{2g(G)}$. Since $g(G_{i_1 j}) = g(G) + 1$, we have

$$P_{2g(G_{i_1 j})} - P_{2g(G_{i_2 j})} = 0 \tag{7.16}$$

Thus

$$\frac{1}{P(G)}(P(G_{i_1 j}) - P(G_{i_2 j}))$$

is of order one in the β expansion. **q.e.d.**

Definition 8. *For a given permutation π let us recursively define the following nested sets of lines of graph G*

$$\mathcal{S}_I \subseteq \mathcal{S}_{I-1} \subseteq \mathcal{S}_{I-2} \subseteq \ldots \subseteq \mathcal{S}_2 \subseteq \mathcal{S}_1 \tag{7.17}$$

as follows.

$$\mathcal{S}_I = \emptyset \tag{7.18}$$

and for $1 \leq i \leq I-1$

$$\mathcal{S}_i = \begin{cases} \mathcal{S}_{i+1} \cup \{\pi_{i+1}\} & \text{if } r(G \setminus \mathcal{S}_{i+1} - \pi_{i+1}) = r(G) \\ \mathcal{S}_{i+1} & \text{if } r(G \setminus \mathcal{S}_{i+1} - \pi_{i+1}) < r(G) \end{cases} \tag{7.19}$$

The complement $\mathcal{S}_k^*$ of $\mathcal{S}_k$ is defined as follows.

$$\mathcal{S}_I^* = \emptyset \tag{7.20}$$

for $k = I$, and

$$\mathcal{S}_k^* = \{\pi_{k+1}, \pi_{k+2}, \ldots, \pi_I\} \setminus \mathcal{S}_k \tag{7.21}$$

for $1 \leq k \leq I-1$.

To illustrate the meaning of $\mathcal{S}_i$ and $\mathcal{S}_i^*$, consider the sequence of lines

$$\pi_I, \pi_{I-1}, \ldots, \pi_{n_1}, \ldots, \pi_{n_2}, \ldots, \pi_{n_3}, \ldots \pi_1$$

ϕ_{i_0} (i.e. for the graph $G - i_0$), it will *a priori* have rank $2g-1$ because there are $2g-1$ independent elements of the vector space buiding up the remaining maps ϕ_i. But I'_{ij} is anti-symmetric. Thus its rank is $2(g-1)$.

with the following properties:

$$\begin{aligned}
&\pi_I, \pi_{I-1}, \ldots, \pi_{n_1+2} \in K_2(G), \\
&\pi_{n_1+1} \notin K_2(G), \\
&\pi_{n_1}, \pi_{n_1-1}, \ldots, \pi_{n_2+2} \in K_2(G - \pi_{n_1+1}), \\
&\pi_{n_2+1} \notin K_2(G - \pi_{n_1+1}), \\
&\pi_{n_2}, \pi_{n_2-1}, \ldots, \pi_{n_3+2} \in K_2(G \setminus \{\pi_{n_1+1}, \pi_{n_2+1}\}), \\
&\pi_{n_3+1} \notin K_2(G \setminus \{\pi_{n_1+1}, \pi_{n_2+1}\}), \\
&\qquad\vdots
\end{aligned} \tag{7.22}$$

Then the sets $\mathcal{S}_i$ read

$$\begin{aligned}
&\mathcal{S}_I = \mathcal{S}_{I-1} = \cdots = \mathcal{S}_{n_1+1} = \emptyset, \\
&\mathcal{S}_{n_1} = \mathcal{S}_{n_1-1} = \cdots = \mathcal{S}_{n_2+1} = \{\pi_{n_1+1}\}, \\
&\mathcal{S}_{n_2} = \mathcal{S}_{n_2-1} = \cdots = \mathcal{S}_{n_3+1} = \{\pi_{n_1+1}, \pi_{n_2+1}\}, \\
&\quad\vdots
\end{aligned} \tag{7.23}$$

One can see from Eq. (7.23) that $r(G \setminus \mathcal{S}_k) = r(G)$ for any k. The meaning of $\mathcal{S}_k^*$ is the following: if one removes any line $l \in \mathcal{S}_k^*$ from the graph $G \setminus \mathcal{S}_k$, the genus of the resulting graph is lower than the genus of G: $r(G \setminus \mathcal{S}_k - l) < r(G)$.

The following lemma is a slight generalization of theorem 1 from section 4.

Lemma 7.2

$$P(G, \theta) = \theta^r \left(\prod_{i=k+1}^{I} \beta_i^{2(L(G \setminus \mathcal{S}_i) - r(G))} \right) P_r(G \setminus \mathcal{S}_k | \vec{\beta}_{\mathcal{S}_k^*} = 1)[1 + \mathcal{O}(\beta)] \tag{7.24}$$

where $r \equiv r(G)$ and $\vec{\beta}_{\mathcal{S}_k^*} = 1$ is a short-hand for $\beta_l = 1, \ \forall\ l \in \mathcal{S}_k^*$.

Proof.

The proof for Eq. (7.24) makes repeated use of lemma 6.2. Let us explain in detail the step from $G \setminus \mathcal{S}_k$ to $G \setminus \mathcal{S}_{k-1}$. From lemma 6.2 we have

$$\begin{aligned}
P_r(G \setminus \mathcal{S}_k \mid \vec{\beta}_{\mathcal{S}_k^*} = 1) &= \beta_k^{2(L(G \setminus \mathcal{S}_k) - r(G))}[1 + \mathcal{O}(\beta)] \times \\
&\times \begin{cases} P_r(G \setminus \mathcal{S}_k - \pi_k | \vec{\beta}_{\mathcal{S}_k^*} = 1) & \text{if } r(G \setminus \mathcal{S}_k - \pi_k) = r(G) \\ P_r(G \setminus \mathcal{S}_k | \vec{\beta}_{\mathcal{S}_k^*} = 1,\, \beta_{\pi_k} = 1) & \text{if } r(G \setminus \mathcal{S}_k - \pi_k) < r(G) \end{cases}
\end{aligned} \tag{7.25}$$

But according to our definition of $\mathcal{S}_i$ and $\mathcal{S}_i^*$ we have:

$$\mathcal{S}_{k-1} = \mathcal{S}_k \cup \{\pi_k\}, \quad \mathcal{S}_{k-1}^* = \mathcal{S}_k^* \quad \text{if } r(G \setminus \mathcal{S}_k - \pi_k) = r(G) \tag{7.26}$$

and

$$\mathcal{S}_{k-1} = \mathcal{S}_k, \quad \mathcal{S}_{k-1}^* = \mathcal{S}_k^* \cup \{\pi_k\} \quad \text{if } r(G \setminus \mathcal{S}_k - \pi_k) < r(G) \tag{7.27}$$

Thus we see that

$$P_r(G \setminus \mathcal{S}_k | \vec{\beta}_{\mathcal{S}_k^*} = 1) = \beta_k^{2(L(G\setminus\mathcal{S}_k)-r(G))} P_r(G \setminus \mathcal{S}_{k-1} | \vec{\beta}_{\mathcal{S}_{k-1}^*} = 1)[1 + \mathcal{O}(\beta)] \tag{7.28}$$

q.e.d.

Lemmas 7.3 and 7.4 given below are direct analogs of lemmas 6.1 and 6.2, respectively.

Lemma 7.3 *Let M be a set of lines of graph G. Let $k \notin M$ be a line of G. Then*

$$X(G, \theta|M) = \begin{cases} X(G-k, \theta|M) & \text{if } L(G-k) = L(G) \\ \alpha_k\, X(G-k, \theta|M) + X(G|M \cup \{k\}) & \text{if } L(G-k) = L(G) - 1 \end{cases} \tag{7.29}$$

where $X(G|M \cup \{k\})$ does not depend on α_k.

Proof.

The proof is a repetition of the proof of lemma 6.1 with the matrix $\mathcal{A}$ replaced by the matrix $\mathcal{A}|_{\alpha_l=0,\ \forall l \in M}$. **q.e.d.**

Lemma 7.4 *Let M be a set of lines of graph G. Let $k \notin M$ be a line of G. Consider the $\theta^{r(G)}$ term in the expansion*

$$X(G, \theta|M) = \sum_{n=0}^{r(G)/2} \theta_i^{2n} X_{2n}(G|M) \tag{7.30}$$

In the expression for $X(G|M)$ let us rescale all Schwinger parameters except the one associated to the line k. Then the following relations hold:

(1) If $L(G-k) = L(G)$, then

$$X_r(\alpha_k, \rho(G-k)|M) = \rho^{L(G)-r(G)} X_r(G|M) \tag{7.31}$$

(2) If $L(G-k) \neq L(G)$ and $r(G-k) < r(G)$, then Eq. (7.31) holds.

(3) If $L(G-k) \neq L(G)$ and $r(G-k) = r(G)$, then

$$X_r(\alpha_k, \rho(G-k)|M) = \rho^{L(G-k)-r(G)} \left[\alpha_k X_r(G-k|M) + \mathcal{O}(\rho)\right] \tag{7.32}$$

Proof.

The proof is completely parallel to that of lemma 6.2. **q.e.d.**

Lemma 7.5 *Let G_1 and G_2 be two graphs. Let $l_1 \in G_1$ and $l_2 \in G_2$ be lines, such that $G_1 - l_1$ and $G_2 - l_2$ differ only by the lines belonging to all trees of $G_1 - l_1$ or $G_2 - l_2$,*

and $r(G_i) = 2 + r(G_i - l_i)$*. Furthermore, assume that* $I_{l_1 n}(G_1) - I_{l_2 n}(G_2) \neq 0$ *only if* $n \in K_2(G_1) \cap K_2(G_2)$*.*

Then:

1) $P_r(G_1) = P_r(G_2)$ *if there are no lines* $n_i \in K_2(G_i)$ *such that* $r(G_i - l_i) = r(G_i - l_i - n_i)$ *and* $L(G_i - l_i) = L(G_i - l_i - n_i) + 1$*.*

2) The relation $P_r(G_1) = P_r(G_2)$ *still holds, even if there is a line* $n_0 \in K_2(G_1) \cap K_2(G_2)$ *with the properties:* $r(G_i - l_i) = r(G_i - l_i - n_0)$*,* $L(G_i - l_i) = L(G_i - l_i - n_i) + 1$ *and* $I_{l_1 n_0}(G_1) = I_{l_2 n_0}(G_2)$*.*

Proof.

Recall that

$$P(G_i, \theta) = \sum_{T, T' \in \mathcal{T}(G_i)} (-)^{T+T'} \det \epsilon[T] \det \epsilon[T'] \det \mathcal{A}[T^* | T'^*] \tag{7.33}$$

Let us show that the contribution to $P_r(G_i)$ comes from trees such that $\mathcal{A}[T^*|T'^*]$ contains the rows and columns corresponding to all the lines in $K_2(G_i)$. Clearly, the intersection matrix with the n-th row and column removed corresponds to the intersection matrix of the graph obtained from the original one by removing the corresponding line n. From the fact that by removing any line from $K_2(G_i)$ the genus decreases, we infer that the rank of the intersection matrix decreases by removing both the row and column associated to any line in $K_2(G_i)$ (see also footnote 21). Thus, if both the row and the column associated to a line in $K_2(G_i)$ are missing from $\mathcal{A}[T^*|T'^*]$, then the power of θ will be less than $\theta^{r(G_i)}$ and the corresponding term will not contribute to $P_r(G_i)$.

Consider next the case when the row associated to a line n in $K_2(G_i)$ is missing from $\mathcal{A}[T^*|T'^*]$, but the column is present. In this situation we perform the Laplace expansion following the column corresponding to the line n. We get

$$\det \mathcal{A}[T^*|T'^*] = \theta \sum (-)^{a_m} I_{nm} \det \mathcal{A}[T^*, \hat{m} | T'^*, \hat{n}] \tag{7.34}$$

where a_m is a number depending on n, m, T and T'. From the assumption that the row associated to a line n in $K_2(G_i)$ is missing from $\mathcal{A}[T^*|T'^*]$ we see that both the row and the column associated to the line n are missing from $\mathcal{A}[T^*, \hat{m}|T'^*, \hat{n}]$. Thus, the maximal power of θ that can come from such a term is $\theta^{r(G_i)-2}$. Combining this with the overall θ we get a power less that $\theta^{r(G_i)}$ and therefore these terms will not contribute to $P_r(G_i)$.

Thus, we conclude that only the terms in which all rows and columns associated to lines in $K_2(G_i)$ are present in $\mathcal{A}[T^*|T'^*]$ can contribute to $P_r(G_i)$.

Let us now show that the lemma is valid. From the assumption, elements that are different between the $\mathcal{A}$ matrices for the graphs G_i involve the intersection of the line l_i and lines of the sets $K_2(G_i)$. Let us take a generic term, $\det \mathcal{A}[T^*|T'^*]$, in the expression for $P(G_i)$. Since we are looking at P_r, all rows and columns associated to lines in $K_2(G_i)$

must be present in $\mathcal{A}[T^*|T'^*]$, as shown above. This implies that neither T nor T' contain those lines. Such trees are trees of $G_i - K_2(G_i)$ and from the assumptions follows that there is a one to one correspondence between the set of these trees for $i = 1$ and $i = 2$.

For convenience, let us label by 1 the line l_i and by n running from 2 to $n_i = card[K_2(G_i)]$ the other lines of $K_2(G_i)$. With this labeling, the difference between the matrices $\mathcal{A}(G_i)$ involves the elements $\mathcal{A}_{1n}$ with n running from 2 to n_i.

We perform the Laplace expansion of our generic term, $\det \mathcal{A}[T^*|T'^*]$, following the first row and the first column. What we get is the following:

$$
\begin{aligned}
\det \mathcal{A}[T^*|T'^*] \ &= \ \theta^2 \sum_{n,m=1}^{N_0} (-)^{a_1} I_{1m} I_{1n} \det \mathcal{A}[T^*, \hat{1}, \hat{m}|T'^*, \hat{1}, \hat{n}] \\
&+\theta^2 \sum_{n=1}^{N_0} \sum_{m=N_0+1}^{I} (-)^{a_2} I_{1n} I_{1m} [\det \mathcal{A}[T^*, \hat{1}, \hat{n}|T'^*, \hat{1}, \hat{m}] + \det \mathcal{A}[T^*, \hat{1}, \hat{m}|T'^*, \hat{1}, \hat{n}]] \\
&+ \ \theta^2 \sum_{n,m=N_0+1}^{I} (-)^{a_3} I_{1m} I_{1n} \det \mathcal{A}[T^*, \hat{1}, \hat{n}|T'^*, \hat{1}, \hat{m}] \qquad (7.35)
\end{aligned}
$$

where $N_0 = card[K_2(G_1) \cap K_2(G_2)]$, a_1, a_2, a_3 are integer numbers depending on m, n, T^* and T'^*. Notice that the last term on the r.h.s. of Eq. (7.35) is the same for both $i = 1$ and $i = 2$ since it does not involve the intersection of l_i with lines in $K_2(G_1) \cap K_2(G_2)$ (by assumption we have $I_{l_1 n}(G_1) - I_{l_2 n}(G_2) \neq 0$ only if $n \in K_2(G_1) \cap K_2(G_2)$).

Now we distinguish the 2 cases stated in the beginning:

1) any line $n \in K_2(G_i)$ is such that the genus of the graph obtained by removing it together with the line l_i equals the genus of the original graph G_i minus 2.

2) there is a line $n_0 \in K_2(G_1) \cap K_2(G_2)$ such that the genus of the graph obtained by removing it together with the line l_i equals the genus of the original graph G_i minus 1.

Let us consider case 1. By assumption we have that

$$
\det \mathcal{A}[T^*, \hat{1}, \hat{n}|T'^*, \hat{1}, \hat{m}] \qquad n,\, m = 2, ..., n_i \qquad (7.36)
$$

comes with at most $\theta^{2g(G_i)-4}$. Together with the θ^2 coefficient, this makes less than the required $\theta^{2g(G_i)}$ for $P_r(G_i)$. Thus, it does not contribute to $P_r(G_i)$.

Let us now look at the second term in Eq. (7.35) for some fixed n between 1 and N_0. Such a term is:

$$
\theta^2 \sum_{m=N_0+1}^{I} (-)^{a_2} I_{1n} I_{1m} \left(\det \mathcal{A}[T^*, \hat{1}, \hat{n}|T'^*, \hat{1}, \hat{m}] + \det \mathcal{A}[T^*, \hat{1}, \hat{m}|T'^*, \hat{1}, \hat{n}] \right) \qquad (7.37)
$$

Now we perform the Laplace expansion for each of the 2 pieces following the row or column n, as appropriate. This operation gives:

$$
\theta^3 \sum_{m=N_0+1}^{I} \sum_{k=1}^{I} (-)^{a_2+a_4} I_{1n} I_{1m} I_{nk} \left(\det \mathcal{A}[T^*, \hat{1}, n, \hat{k}|T'^*, \hat{1}, \hat{m}, \hat{n}] + (\hat{m} \longleftrightarrow \hat{k}) \right) \qquad (7.38)
$$

where a_4 depends on k, m, n, T^*, T'^*. Each of these minors are otained from the original matrix by removing at least the first and n-th row and column. But by assumption such a determinant brings at most $\theta^{2g(G_i)-4}$. Together with the prefactor this makes less than $\theta^{2g(G_i)}$ required for $P_r(G_i)$ and thus it will not contribute.

We are therefore left with:

$$\det \mathcal{A}[T^*|T'^*] = \theta^2 \sum_{n,m=N_0+1}^{I} (-)^{a_3} I_{1m} I_{1n} \det \mathcal{A}[T^*, \hat{1}, \hat{m}|T'^*, \hat{1}, \hat{n}] \tag{7.39}$$

Since the sum does not run over the intersections between l_i and $K_2(G_i)$, these terms are present for both $i = 1$ and $i = 2$..

Consider now case 2. As in case 1, the last term in Eq. (7.35) is the same for both $P_r(G_1)$, but we can get nontrivial contributions to P_r from the first two terms. In particular, we have the terms:

$$\det \mathcal{A}[T^*\hat{1}, \hat{n}_0|T'^*, \hat{1}, \hat{n}_0] \tag{7.40}$$

$$\det \mathcal{A}[T^*\hat{1}, \hat{n}_0|T'^*, \hat{1}, \hat{k}], \quad k = 2, ..., n_i, \quad k \neq n_0 \tag{7.41}$$

$$\det \mathcal{A}[T^*\hat{1}, \hat{n}_0|T'^*, \hat{1}, \hat{m}], \quad m = n_i + 1, ..., I \tag{7.42}$$

But, because $n_0 \in K_2(G_1) \cap K_2(G_2)$ and $I_{l_1 n_0}(G_1) = I_{l_2 n_0}(G_2)$ the first and the last term are present both in $P_r(G_1)$ and $P_r(G_2)$. For the second term, by performing the Laplace expansion as in Eq. (7.38) we find that it does not contribute to $P_r(G_i)$.

We conclude therefore that the second statement of the lemma holds as well. **q.e.d.**

Lemma 7.6 *Let γ_{k_0} be the smallest subgraph of G that has $j(\gamma_{k_0}) = 1$ (let i_0 be the external line such that $c_{G_{i_0 j}}(\gamma_{k_0}) > c(\gamma_{k_0})$) and let γ_{k_r} be the smallest subgraph of G that has $c_{G_{\pm rj}}(\gamma_{k_r}) > c_G(\gamma_{k_r})$. We assume that $\gamma \subset \gamma_{k_0}$.*

Then, $G_{rj} \setminus \mathcal{S}_k(G_{rj})$, $\forall\ j = 1, ..., E$ satisfy the conditions of lemma 7.5 for $k < k_0$.

Proof.

Let i_0 be the external line such that $c_{G_{i_0 j}}(\gamma_{k_0}) > c(\gamma_{k_0})$. The line l_j in lemma 7.5 is the line joining (the line carrying momentum) r and external line j. If $k < k_0$ then $j(G_{rj} \setminus \mathcal{S}_k(G_{rj}) \setminus \mathcal{S}^*_k(G_{rj})) = 0$. By construction, removing any one of the lines in $\mathcal{S}^*_k(G_{rj})$ changes the genus of $G_{rj} \setminus \mathcal{S}_k(G_{rj})$ Therefore, all the lines in $\mathcal{S}^*_k(G_{rj})$ belong to $K_2(G_{rj} \setminus \mathcal{S}_k(G_{rj}))$.

I. $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$.

Proof by induction:

1) By assumption, $G_{rj} - l_j = G$ are identical. Thus, the first step in the induction needs no further proof.

2) Prove $k \Rightarrow k-1$, i.e. assume that $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ and prove the same for $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$. For the general implication, we distinguish two possible mutually exclusive situations:

a) the line π_k in the definition of $\mathcal{S}_{k-1}(G_{rj})$ belongs to all the trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$

b) the line π_k in the definition of $\mathcal{S}_{k-1}(G_{rj})$ belongs to a loop in $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$

These two cases are mutually exclusive because, by assumption, all the graphs $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ which implies that we cannot have case **a** for some j-s and case **b** for the rest.

For each of these situations we distinguish two subcases.

For the case **a**, the line π_k can belong, for some j-s, to all the trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj})$, while for the rest of j-s the line π_k belongs to all the trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$. In the first subcase $\mathcal{S}_{k-1}(G_{rj}) = \mathcal{S}_k(G_{rj}) \cup \{\pi_k\}$ and $\mathcal{S}^*_{k-1}(G_{rj}) = \mathcal{S}^*_k(G_{rj})$, while in the second subcase $\mathcal{S}_{k-1}(G_{rj}) = \mathcal{S}_k(G_{rj})$ and $\mathcal{S}^*_{k-1}(G_{rj}) = \mathcal{S}^*_k(G_{rj}) \cup \{\pi_k\}$. It follows then that in the first subcase π_k does not appear in $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj})$, while the second subcase π_k has the property that l_j does not belong to a loop in $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - \pi_k$ which implies that π_k does not belong to a loop in $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$. Thus, we get that $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$ differ only by lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$.

For the case **b**, the line π_k can belong either to $\mathcal{S}^*_{k-1}(G_{rj})$ or to $\mathcal{S}_{k-1}(G_{rj})$.

In the first subcase, since π_k belongs to a loop in $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ it follows that $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - \pi_k$ contains l_j in a loop. The fact that $\pi_{k-1} \in \mathcal{S}^*_{k-1}(G_{rj})$ (i.e. by removing π_k an independent cycle on $\Sigma(G_{rj})$ is not associated to any loop in $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - \pi_k$) implies that π_k belongs to a loop in $G_{rj} \setminus \mathcal{S}_k(G_{rj})$. From the assumption that $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ follows that the line π_k belongs to a loop in $G_{rj} \setminus \mathcal{S}_k(G_{rj})$ for any j. From the addition of cycles now follows that l_j belongs to a loop in $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - \pi_k$ for any j.[22] Thus, we get that $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$ for any j.

Let us point out that the two subcases are again mutually exclusive because, as shown above, if the first subcase is realized, then it is realized for all $j = 1, ..., E$.

In the second subcase, since $\pi_k \in \mathcal{S}_{k-1}(G_{rj})$, π_k does not appear in $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj})$. Furthermore, since the two subcases are mutually exclusive, this subcase is either realized for all $j = 1, ..., E$ or is not realized at all. It follows therefore that in the case **b**, $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$ differ only by lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_{k-1}(G_{rj}) - l_j$, as well.

II. $I_{l_1 n}(G_{rj_1} \setminus \mathcal{S}_k(G_{rj_1})) - I_{l_2 n}(G_{rj_2} \setminus \mathcal{S}_k(G_{rj_2})) \neq 0$ only if $n \in \cap_{j=1}^{E} K_2(G_{rj} \setminus \mathcal{S}_k(G_{rj}))$.

We can construct the intersection matrix such that the intersections between l_j and the lines of γ_{k_0} are the same for any j.[23] From the fact that $j(G_{rj} \setminus \mathcal{S}_k(G_{rj}) \setminus \mathcal{S}^*_k(G_{rj})) = 0$ for any $k < k_0$ we see that the difference in the intersection matrices for different j-s involve the intersection of l_j and other lines in $\mathcal{S}^*_k(G_{rj})$. But, as we have shown before, all the lines in $\mathcal{S}^*_k(G_{rj})$ belong to $K_2(G_{rj} \setminus \mathcal{S}_k(G_{rj}))$. Furthermore, we have shown above

[22] Recall that $\pi_{k-1} \in \mathcal{S}^*_{k-1}(G_{rj})$ also implies that $\mathcal{S}_k(G_{rj}) = \mathcal{S}_{k-1}(G_{rj})$

[23] We take $I_{l_* n}(G_{rj}) = I_{l_* n}(G_{ri_0}) + I_{l_* n}(G_{i_0} j)$

that $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j$. Because of equality of momenta there is no intersection between two lines that belong to a single loop. Thus we have that $I_{l_1 n}(G_{rj_1} \setminus \mathcal{S}_k(G_{rj_1})) - I_{l_2 n}(G_{rj_2} \setminus \mathcal{S}_k(G_{rj_2})) \neq 0$ only if $n \in \cap_{j=1}^{E} K_2(G_{rj} \setminus \mathcal{S}_k(G_{rj}))$.

III. If $\gamma_{k_r} \subset \gamma_k = G_{rj} \setminus \mathcal{S}_k(G_{rj}) \setminus \mathcal{S}^*_k(G_{rj})$ then any line in $n \in \mathcal{S}^*_k(G_{rj})$ is such that $r(G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j) \neq r(G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j - n)$ because the lines violating this condition belong to γ_{k_r}[24]

IV. If $\gamma_k = G_{rj} \setminus \mathcal{S}_k(G_{rj}) \setminus \mathcal{S}^*_k(G_{rj}) \subset \gamma$ then there is a line $n \in \mathcal{S}^*_k(G_{rj})$ such that $r(G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j) = r(G_{rj} \setminus \mathcal{S}_k(G_{rj}) - l_j - n)$. But, as we said before, intersection matrix can be constructed such that the intersections between l_j and the lines of γ_{k_0} are the same for any j. Thus the second statement of lemma 7.5 is satisfied.

This shows that the graphs $G_{rj} \setminus \mathcal{S}_k(G_{rj})$ for $k < k_0$ satisfy the assumptions of lemma 7.5. **q.e.d.**

Definition 9.*Following the model of $\mathcal{S}$ described in Definition 8, we define the following nested sets:*

$$\mathcal{Z}_{k_0} \subset \mathcal{Z}_{k_0-1} \subset \ldots \subset \mathcal{Z}_2 \subset \mathcal{Z}_1 \tag{7.43}$$

$$\mathcal{Z}_{k_0} = \emptyset \tag{7.44}$$

$$\mathcal{Z}_i = \begin{cases} Z_{i+1} \cup \{\pi_{i+1}\} & \text{if } \pi_{k_0} \text{ does not belongs to a loop in } G \setminus (S_{i+1} \setminus \mathcal{Z}_{i+1}) - \pi_{i+1} - l \\ Z_{i+1} & \text{if } \ \pi_{k_0} \text{ belongs to a loop in } G \setminus (S_{i+1} \setminus \mathcal{Z}_{i+1}) - \pi_{i+1} - l \end{cases} \tag{7.45}$$

where l is some fixed line in G.

We define also

$$\mathcal{S}'_i = \mathcal{S}_i \setminus Z_i \qquad \mathcal{S}'^*_i = \mathcal{S}^*_i \setminus \mathcal{Z}_i \tag{7.46}$$

where $\mathcal{S}_i$ and $\mathcal{S}^*_i$ were defined before.

Remark. From the above definition it follows that $\mathcal{S}'_i \subseteq \mathcal{S}_i$ with equality for all $i \geq k_0$ and that $\mathcal{Z}_{k_0-1} = \{\pi_{k_0}\}$.

Now we would like to prove an analog of lemma 7.6 for the sets $\mathcal{S}'$. The proof is similar to that of lemma 7.6 so we will not go into much details. However, there is an important difference which will be pointed out.

Lemma 7.7 *Let γ_{k_0} be the smallest subgraph of G that has $j(\gamma_{k_0}) = 1$ (let i_0 be the external line such that $c_{G_{i_0 j}}(\gamma_{k_0}) > c(\gamma_{k_0})$) and let γ_{k_r} be the smallest subgraph of G that has $c_{G_{\pm rj}}(\gamma_{k_r}) > c_G(\gamma_{k_r})$. We assume that $\gamma \subset \gamma_{k_0}$.*

Then, $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_{k_0}$, $\forall\ j = 1, ..., E$ satisfy the conditions of lemma 7.5 for $k < k_0$ where in the definition of the sets $\mathcal{Z}$ the fixed line l is chosen to be l_j.

[24]in detail, this goes as follows: introducing l_j adds 2 cycles to the graph G (say a and b). One of them (say a) is disconnected by removing l_j. From the fact that $\gamma_{k_r} \subset \gamma_k$ is such that $c_{G_{\pm rj}}(\gamma_{k_r}) > c_G(\gamma_{k_r})$ follows that this subgraph wraps the other cycle (b). This implies that there is no line in $\mathcal{S}^*_k(G_{rj})$ supporting cycle (b).

Proof.

I. $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ for any $\pi_s \in \mathcal{Z}_k$.

The proof, just like for lemma 7.6, is by induction.

The first step follows immediately from lemma 7.6. More precisely, using the remark that $\mathcal{Z}_{k_0-1}(G_{rj}) = \{\pi_{k_0}\}$ follows that $G_{rj} \setminus \mathcal{S}'_{k_0-1}(G_{rj}) - \pi_{k_0} = G_{rj} \setminus (\mathcal{S}_{k_0-1}(G_{rj}) - \pi_{k_0}) - \pi_{k_0} = G_{rj} \setminus \mathcal{S}_{k_0-1}(G_{rj})$ and the last graph satisfies the desired relation as implied by lemma 7.6.

For the general implication we start by making two simple observations:

1. we notice that, since by definition the elements of the set $\mathcal{Z}_k$ have the property that by removing any one of them, $\pi_l \in \mathcal{Z}_k$, the line π_{k_0} does not belong to a loop in $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_l$ it follows that in $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_l$ none of the elements of $\mathcal{Z}_k$ belongs to a loop.

2. we notice that if $\pi_k \in \mathcal{Z}_{k-1}$ then π_k necessarily belongs to a loop in $G_{rj} \setminus \mathcal{S}'_k(G_{rj})$.

With these two observations we follow the analysis in lemma 7.6 and assume that $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - l_j$ differ only by lines belonging to all trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - l_j$ and show the same property for $G_{rj} \setminus \mathcal{S}'_{k-1}(G_{rj}) - l_j$. We distinguish two main situations:

A) $\pi_k \notin \mathcal{Z}_{k-1}$

B) $\pi_k \in \mathcal{Z}_{k-1}$

In case **A** the analysis is identical to the one in lemma 7.6 and we will not repeat it.

In case **B** we again distinguish the two situations from lemma 7.6:

a) the line π_k belongs to all the trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - l_j$

b) the line π_k belongs to a loop in $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - l_j$

Unlike lemma 7.6, case **a** has no subcases due to the observations in the beginning. In particular, π_k cannot belong to all trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj})$ because in $G_{rj} \setminus \mathcal{S}'_k(G_{rj})$ the line π_{k_0} belongs to a loop while in $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_k$ it does not.

In the case **b** we have the same two subcases as in lemma 7.6: the line π_k can belong either to $\mathcal{S}^*_{k-1}(G_{rj})$ or to $\mathcal{S}_{k-1}(G_{rj})$. The analysis is similar and we will not repeat it.

Collecting everything we see that we have proven the desired result. Another important outcome of this discussion is that if $\pi_k \in \mathcal{Z}_{k-1}$ there are three mutually exclusive possibilities (case **a** and the two subcases of case **b**). This fact, that if one of them is realized for some j then it is realized for all $j = 1, ..., E$, will be important in theorem 4.

II. $I_{l_1 n}(G_{rj_1} \setminus \mathcal{S}'_k(G_{rj_1}) - \pi_s) - I_{l_2 n}(G_{rj_2} \setminus \mathcal{S}'_k(G_{rj_2}) - \pi_s) \neq 0$ only if $n \in \cap_{j=1}^{E} K_2(G_{rj} \setminus \mathcal{S}'_k(G_{rj}))$ where $\pi_s \in \mathcal{Z}_k(G_{rj})$.

This relation follows from the observation in the beginning that in $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s$ none of the lines in $\mathcal{Z}_k$ belongs to a loop. Thus, as far as the intersection matrix is

concerned, this graph is equivalent to $G_{rj} \setminus \mathcal{S}_k(G_{rj})$. This latter graph has the desires property as shown in lemma 7.6.

Points **III.** and **IV.** are identical to those in lemma 7.6 and we will not repeat the arguments. **q.e.d.**

Theorem 4 *Let γ_{k_0} be the subgraph considered in theorem 2. The leading term in the expansion of*

$$\sum_j r_i Q_{ij} p_j \tag{7.47}$$

in β, after the summation over all $j \in \mathcal{E}(G)$ and using the momentum conservation for $p_j,\ \ j \in \mathcal{E}(G)$, is of the form

$$\left(\prod_{l=k_1}^{I} \beta_l^{-2} \right) R_1(\beta_1, \beta_2, \ldots, \beta_{k_1-1}) \tag{7.48}$$

for some $k_1 \geq k_0$, where R_1 is a polynomial in variables $\beta_1, \beta_2, \ldots, \beta_{k_1-1}$.

Proof.

Let us begin by pointing out that if $j(G) = 1$ and $r(G_{\pm rj}) = r(G)$ for all $j = 1, ..., E$, then the theorem is trivially satisfied because each term in the sum in Eq. (7.47) is of order β^0. We thus assume that $r(G_{\pm rj}) > r(G)$.

As in theorem 2, let γ_{k_0} be the smallest subgraph of G from Eq. (5.4) that has $j(\gamma_{k_0}) = 1$. This implies that if we join any two external lines i, j from $\mathcal{E}(G)$, we have $c_{G_{ij}}(\gamma_{k_0} - \pi_{k_0}) = c(\gamma_{k_0} - \pi_{k_0})$. Using the decomposition in lemma 6.4, let i_0 be a line in $\mathcal{E}_1(\gamma_{k_0})$ such that $c_{G_{i_0 j}}(\gamma_{k_0}) > c(\gamma_{k_0})$ for all $j \in \mathcal{E}_2(\gamma_{k_0})$.

Consider now a pair of lines $i_1, i_2 \in \mathcal{E}'(G)$ attached to a line $\bar{\imath}$ with $deg(\bar{\imath}) > 0$. Let γ_{k_r} be the smallest subgraph from Eq. (5.4) for which $c_{G_{i'j}}(\gamma_{k_r}) > c(\gamma_s)$ or $c_{G_{i''j}}(\gamma_{k_r}) > c(\gamma_s)$. Let us assume, for concreteness, that $c_{G_{i'j}}(\gamma_{k_r}) > c(\gamma_s)$ and that the momentum carried by the line i' is r. To emphasize this, let us denote the graph $G_{i'j}$ by G_{rj}. There are two possibilities:

(1) $\gamma_{k_0} \subseteq \gamma_{k_r}$ ($k_0 \leq k_r$)

(2) $\gamma_{k_r} \subset \gamma_{k_0}$ ($k_r < k_0$)

In the first case it is easy to see that

$$\frac{P(G_{\pm rj})}{P(G)} = \prod_{n=k_r}^{I} \frac{1}{\beta_n^2} \qquad \frac{P(G_{i_0 j})}{P(G)} = \prod_{n=k_0}^{I} \frac{1}{\beta_n^2} \quad i_0 \neq j$$

while from the assumption that $\gamma_{k_0} \subset \gamma_{k_r}$ we see that $k_0 \leq k_r$. Therefore, the term $p_{i_0} p_j$, $i_0 \neq j$ which exists in the exponent will regulate whatever we bring down from $r_i p_j$.

Imagine now that the lines carrying momenta $\pm r$ lie in the interior of γ_{k_0} (i.e. $\gamma_{k_r} \subset \gamma_{k_0}$). For each one of them, say the one carrying momentum $+r$, we have the following

contribution to the $r \cdot p$ term:

$$Q(+r,p) = \frac{r \cdot \sum_{i=1}^{E} P(G_{rj})\ p_j}{P(G)} \quad . \tag{7.49}$$

Using lemma 7.2 (Eq. (7.24)) we can identify the parts that cancel in the β expansion of the $r \cdot p$ terms. Let us write it for $k = k_0$ and insert it in the previous equation. We get:

$$Q(+r,p) = r \cdot \left(\prod_{i=k_0+1}^{I} \frac{1}{\beta_i^2} \right) \sum_{j=1}^{E} \frac{P_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1)}{P_r(G \setminus \mathcal{S}_{k_0}(G) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G)} = 1)} p_j [1 + \mathcal{O}(\beta)] \tag{7.50}$$

As explained in footnote 18, the permutation used for the graphs G_{rj} are constrained to have the line joining r and j as π_{I+1} and β_{I+1} is set to 1. We can now write the numerator for each term in the sum using lemmas 6.1 and 6.2, to get

$$\begin{aligned} P_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1) &= \beta_{k_0}^{2(L(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj})) - r(G))} \times \qquad (7.51) \\ \Big(P_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) - \pi_{k_0} | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1) &+ X_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1 | \pi_{k_0}) \Big) \end{aligned}$$

We remind the reader that, as introduced in Eq. (6.14), the notation $X_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1 | \pi_{k_0})$ is defined to be $P_r(G_{rj} \setminus \mathcal{S}_{k_0}(G_{rj}) | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj})} = 1)$ with $\alpha_{\pi_{k_0}} = 0$. Using lemmas 7.5 and 7.6 we have that

$$P_r(G_{rj_1} \setminus \mathcal{S}_{k_0}(G_{rj_1}) - \pi_{k_0} | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj_1})} = 1) = P_r(G_{rj_2} \setminus \mathcal{S}_{k_0}(G_{rj_2}) - \pi_{k_0} | \vec{\beta}_{\mathcal{S}^*_{k_0}(G_{rj_2})} = 1) \tag{7.52}$$

for all $j_1,\ j_2 = 1, \dots, E$. Therefore, they all cancel[25] by momentum conservation:

$$\sum_{j=1}^{E} p_j = 0 \tag{7.53}$$

while the surviving terms, using the quantities introduced in Definition 9 and the remark following it, are

$$Q(+r,p) = r \cdot \left(\prod_{i=k_0}^{I} \frac{1}{\beta_i^2} \right) \sum_{j=1}^{E} \frac{X_r(G_{rj} \setminus \mathcal{S}'_{k_0-1}(G_{rj}) | \vec{\beta}_{\mathcal{S}'^*_{k_0-1}(G_{rj})} = 1 | \mathcal{Z}_{k_0-1})}{P_r(G \setminus \mathcal{S}_{k_0-1}(G) | \vec{\beta}_{\mathcal{S}^*_{k_0-1}(G)} = 1)} p_j [1 + \mathcal{O}(\beta)] \tag{7.54}$$

Thus, we need an expansion of $X_r(G_{rj} \setminus \mathcal{S}'_{k_0-1}(G_{rj}) | \vec{\beta}_{\mathcal{S}'^*_{k_0-1}(G_{rj})} = 1 | \mathcal{Z}_{k_0-1})$. It turns out that it is more useful to analyze the whole sum in the numerator of Eq. (7.54). As it has became obvious from Eq. (7.52) and Eq. (7.53), for $k < k_0$ the leading term in the expansion of the numerator of Eq. (7.54) does not come from $P_r(G_{rj} \setminus \mathcal{S}_k(G_{rj}))$

[25] Let us point out that this cancelation may occur even before reaching the line π_{k_0}. Here we treat the most singular scenario.

because, according to lemmas 7.5 and 7.6, such terms cancel when the sum over momenta is performed. Furthermore, lemmas 7.5 and 7.7 implies that the leading term comes from graphs that contain the line π_{k_0} in a loop (if π_{k_0} does not belong to a loop in $G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s$, then X_r can be interpreted as a P_r and from lemma 7.5 they cancel by momentum conservation.).

Let us now find the leading term in the expansion of the numerator of Eq. (7.54). The claim is that it is given by

$$\sum_{j=1}^{E} X_r(G_{rj} \setminus \mathcal{S}'_{k_0-1}(G_{rj})|\vec{\beta}_{\mathcal{S}'^*_{k_0-1}(G_{rj})} = 1|\mathcal{Z}_{k_0-1})p_j = \prod_{i=s+1}^{k_0-1} \beta_i^{2[L(G_{rj}\setminus\mathcal{S}'_i(G_{rj}))-r(G_{rj})]} \times$$
$$\times \sum_{j=1}^{E} X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}))|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s)p_j[1+\mathcal{O}(\beta)] \quad . \tag{7.55}$$

Here we can take $\beta_i^{2[L(G_{rj}\setminus\mathcal{S}'_i(G_{rj}))-r(G_{rj})]}$ out of the sum because both $r(G_{rj}) = r(G)+2$ and $L(G_{rj} \setminus \mathcal{S}'_i(G_{rj}))$ are the same for all j as will be shown.

The proof for this relation uses lemma 7.4. Let us explain in detail the step from $G_{rj} \setminus \mathcal{S}'_s(G_{rj})$ to $G_{rj} \setminus \mathcal{S}'_{s-1}(G_{rj})$. Using lemma 7.4 we write:

$$\sum_{j=1}^{E} X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}))|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s)p_j = \beta_s^{2[L(G_{rj}\setminus\mathcal{S}'_s(G_{rj}))-r(G_{rj})]} \times \tag{7.56}$$
$$\sum_{j=1}^{E} \begin{cases} X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s)p_j & \begin{array}{l}\text{if } r(G\setminus\mathcal{S}_s - \pi_s) = r(G) \\ \text{and } \pi_{k_0} \text{ belongs to a} \\ \text{loop in } G_{rj}\setminus\mathcal{S}'_s(G_{rj}) - \pi_s,\end{array} \\ X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj})|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1, \beta_{\pi_s} = 1|\mathcal{Z}_s)p_j & \begin{array}{l}\text{if } r(G\setminus\mathcal{S}_s - \pi_s) < r(G) \\ \text{and } \pi_{k_0} \text{ belongs to a} \\ \text{loop in } G_{rj}\setminus\mathcal{S}'_s(G_{rj}) - \pi_s,\end{array} \\ X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj})|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s, \pi_s)p_j & \begin{array}{l}\text{if } r(G\setminus\mathcal{S}_s - \pi_s) < r(G) \\ \text{and } \pi_{k_0} \text{ does not belong to} \\ \text{a loop in } G_{rj}\setminus\mathcal{S}'_s(G_{rj}) - \pi_s,\end{array} \\ \begin{array}{c} X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s)p_j + \\ X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj})|\vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1|\mathcal{Z}_s, \pi_s)p_j \end{array} & \begin{array}{l}\text{if } r(G\setminus\mathcal{S}_s - \pi_s) = r(G) \\ \text{and } \pi_{k_0} \text{ does not belongs to} \\ \text{a loop in } G_{rj}\setminus\mathcal{S}'_s(G_{rj}) - \pi_s\end{array} \end{cases}$$

where we omitted $[1+\mathcal{O}(\beta)]$ on the r.h.s.

Using the definitions of $\mathcal{S}, \mathcal{S}', \mathcal{Z}$ and their starred partners, we can assemble the arguments of X in the four cases in Eq. (7.56) as follows:

case 1. if $r(G_{rj} \setminus \mathcal{S}'_s - \pi_s) = r(G_{rj})$ and π_{k_0} belongs to a loop in $G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s$ then $\mathcal{S}_{s-1} = \mathcal{S}_s \cup \{\pi_s\}$, $\mathcal{S}^*_{s-1} = \mathcal{S}^*_s$ and $\mathcal{Z}_{s-1} = \mathcal{Z}_s$. Thus we have

$$\mathcal{S}'_{s-1} = \mathcal{S}_{s-1} \setminus \mathcal{Z}_{s-1} = (\mathcal{S}_s \cup \{\pi_s\}) \setminus \mathcal{Z}_s = \mathcal{S}'_s \cup \{\pi_s\} \tag{7.57}$$

$$\mathcal{S}'^{*}_{s-1} = \mathcal{S}^{*}_{s-1} \setminus \mathcal{Z}_{s-1} = \mathcal{S}^{*}_{s} \setminus \mathcal{Z}_{s} = \mathcal{S}'^{*}_{s} \tag{7.58}$$

case 2. if $r(G_{rj} \setminus \mathcal{S}'_s - \pi_s) < r(G_{rj})$ and π_{k_0} belongs to a loop in $G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s$ then $\mathcal{S}_{s-1} = \mathcal{S}_s$, $\mathcal{S}^*_{s-1} = \mathcal{S}^*_s \cup \{\pi_s\}$ and $\mathcal{Z}_{s-1} = \mathcal{Z}_s$. Thus we have

$$\mathcal{S}'_{s-1} = \mathcal{S}_{s-1} \setminus \mathcal{Z}_{s-1} = \mathcal{S}_{s} \setminus \mathcal{Z}_{s} = \mathcal{S}'_{s} \tag{7.59}$$

$$\mathcal{S}'^{*}_{s-1} = \mathcal{S}^{*}_{s-1} \setminus \mathcal{Z}_{s-1} = (\mathcal{S}^{*}_{s} \cup \{\pi_s\}) \setminus \mathcal{Z}_{s} = \mathcal{S}'^{*}_{s} \cup \{\pi_s\} \tag{7.60}$$

case 3. if $r(G_{rj} \setminus \mathcal{S}'_s - \pi_s) < r(G_{rj})$ and π_{k_0} does not belongs to a loop in $G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s$ then $\mathcal{S}_{s-1} = \mathcal{S}_s$, $\mathcal{S}^*_{s-1} = \mathcal{S}^*_s \cup \{\pi_s\}$ and $\mathcal{Z}_{s-1} = \mathcal{Z}_s \cup \{\pi_s\}$. Thus, we have

$$\mathcal{S}'_{s-1} = \mathcal{S}_{s-1} \setminus (\mathcal{Z}_{s} \cup \{\pi_s\}) = \mathcal{S}_{s} \setminus \mathcal{Z}_{s} = \mathcal{S}'_{s} \tag{7.61}$$

$$\mathcal{S}'^{*}_{s-1} = \mathcal{S}^{*}_{s-1} \setminus \mathcal{Z}_{s-1} = (\mathcal{S}^{*}_{s} \cup \{\pi_s\}) \setminus (\mathcal{Z}_{s} \cup \{\pi_s\}) = \mathcal{S}'^{*}_{s} \tag{7.62}$$

This situation is covered by case **a** and first subcase of case **b** in the proof of lemma 7.7. As pointed out before, these cases are mutually exclusive, i.e. each one either occurs for all $j = 1, ..., E$ or it does not accur at all.

case 4. We begin by studying the first term. If π_{k_0} does not belong to a loop in $G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s$ then, from the observation **1.** in the proof of lemma 7.7, we get that

$$X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}) - \pi_s | \vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1 | \mathcal{Z}_s) = P(G_{rj} \setminus \mathcal{S}'_s(G_{rj}) \setminus \mathcal{Z}_s - \pi_s | \vec{\beta}_{\mathcal{S}'^*_s(G_{rj})} = 1) \tag{7.63}$$

This situation is covered by second subcase of case **b** in the proof of lemma 7.7. Since this subcase is exclusive with respect to the other cases in that lemma it follows that, if it occurs, it will occur for all $j = 1, ..., E$. Therefore, using the lemmas 7.7 and 7.5, we find that the first term will cancel upon summing over momenta.

For the second term we have: $\mathcal{S}_{s-1} = \mathcal{S}_s \cup \{\pi_s\}$, $\mathcal{S}^*_{s-1} = \mathcal{S}^*_s$ and $\mathcal{Z}_{s-1} = \mathcal{Z}_s \cup \{\pi_s\}$. This implies that

$$\mathcal{S}'_{s-1} = \mathcal{S}_{s-1} \setminus \mathcal{Z}_{s-1} = (\mathcal{S}_{s} \cup \{\pi_s\}) \setminus (\mathcal{Z}_{s} \cup \{\pi_s\}) = \mathcal{S}_{s} \setminus \mathcal{Z}_{s} = \mathcal{S}'_{s} \tag{7.64}$$

$$\mathcal{S}'^{*}_{s-1} = \mathcal{S}^{*}_{s-1} \setminus \mathcal{Z}_{s-1} = \mathcal{S}^{*}_{s} \setminus (\mathcal{Z}_{s} \cup \{\pi_s\}) = \mathcal{S}^{*}_{s} \setminus \mathcal{Z}_{s} = \mathcal{S}'^{*}_{s} \tag{7.65}$$

In combining all the above cases we have to use again lemma 7.7. In particular, we use that the union of cases 1 and 2, case 3 and case 4 in Eq. (7.56) are mutually exclusive together with the fact that $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ for all $j = 1, ..., E$ and any $\pi_s \in \mathcal{Z}_k(G_{rj})$. These two properties allow us to pull out of the sum the factor $\beta_s^{2[L(G_{rj} \setminus \mathcal{S}'_s(G_{rj})) - r(G_{rj})]}$.[26] Therefore,

[26] The fact that $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ are identical up to lines belonging to all trees of $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ for all $j = 1, ..., E$ and any $\pi_s \in \mathcal{Z}_k(G_{rj})$ implies, in particular, that the number of loops of these graphs is the same for all j. Gluing l_j to $G_{rj} \setminus \mathcal{S}'_k(G_{rj}) - \pi_s - l_j$ we can have two situations: an extra loop is produced (situation covered by the first subcase of case **b** of lemma 7.7) or it is not (situation covered by case **a** of lemma 7.7). Since these cases are mutually exclusive, it follows that the numbers of loops of $G_{rj} \setminus \mathcal{S}'_k(G_{rj})$ are the same for all $j = 1, ..., E$.

Eq. (7.56) becomes:

$$\sum_{j=1}^{E} X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj}))|\vec{\beta}_{\mathcal{S'}^*_s(G_{rj})} = 1|\mathcal{Z}_s)p_j = \beta_s^{2(L(G_{rj}\setminus\mathcal{S}'_s(G_{rj}))-r(G_{rj}))} \times$$
$$\sum_{j=1}^{E} X_r(G_{rj} \setminus \mathcal{S}'_{s-1}(G_{rj}))|\vec{\beta}_{\mathcal{S'}^*_{s-1}(G_{rj})} = 1|\mathcal{Z}_{s-1})p_j \tag{7.66}$$

Replacing all this in the ratio Eq. (7.54) we get:

$$\sum_{j=1}^{E} \frac{X_r(G_{rj} \setminus \mathcal{S}'_{k_0-1}(G_{rj})|\vec{\beta}_{\mathcal{S'}^*_{k_0-1}(G_{rj})} = 1|\mathcal{Z}_{k_0-1})}{P_r(G \setminus \mathcal{S}_{k_0-1}(G)|\vec{\beta}_{\mathcal{S}^*_{k_0-1}(G)} = 1)} p_j = \tag{7.67}$$
$$\left(\prod_{i=s+1}^{k_0-1} \beta_i^{2(L(G_{rj}\setminus\mathcal{S}'_i(G_{rj}))-L(G-S_i(G))-2)}\right) \sum_{j=1}^{E} \frac{X_r(G_{rj} \setminus \mathcal{S}'_s(G_{rj})|\vec{\beta}_{\mathcal{S'}^*_s(G_{rj})} = 1|\mathcal{Z}_s)}{P_r(G \setminus \mathcal{S}_s(G)|\vec{\beta}_{\mathcal{S}^*_s(G)} = 1)} p_j$$

As pointed out before, from the definition of $\mathcal{S}_i$ and $\mathcal{S}'_i$ we see that $\mathcal{S}'_i \subseteq \mathcal{S}_i$ and the number of loops of $G_{rj} \setminus \mathcal{S}'_i(G_{rj})$ is always larger by at least 2 than the number of loops of $G \setminus \mathcal{S}_i(G)$ for all $i < k_0$ (1 loop from the joining (rj) and at least another loop from never letting the line π_{k_0} belong to all trees of $G_{rj} \setminus \mathcal{S}'_i(G_{rj})$).

Eq. (7.67) and the comment following it are valid for any $s = k_r, ..., k_0 - 1$ where π_{k_r} has the property that by removing it the line on which the derivatives act is disconnected from all loops. On the other hand we know that if a subgraph of G does not contain this line k_r in a loop, then its cycle number computed with respect to any joining of $\pm r$ with external lines is the same as computed with respect to the original graph G. Therefore, we conclude that the β expansion of the terms linear in r cannot be more singular than:

$$f(\beta_1, ..., \beta_{k_0-1}) \prod_{i=k_0}^{I} \frac{1}{\beta_i^2} \tag{7.68}$$

where γ_{k_0} is the smallest subgraph of G in the permutation π that has $j(\gamma_{k_0}) = 1$. At the same time, the leading term independent of r in the same permutation π is

$$f(\theta, p) \prod_{i=k_0}^{I} \frac{1}{\beta_i^2} \ . \tag{7.69}$$

This shows that, for the non-exceptional momenta, the $r \cdot p$ terms are always regulated by the p^2 terms.   **q.e.d.**

Remark. From the details of the proof of theorem 4 follows that it is not necessary that the line carrying external momentum r be associated to lines in the set $\mathcal{E}'(G)$. All we have used is that $\sum_{j\neq r} p_j = 0$. Therefore, r can denote any fixed line as long as the sum over the momenta of the lines different from r appearing in Eq. (7.47) vanishes.

Lemma 7.8 *The leading term in the expansion of $\tilde{Q}(p, \alpha, \Theta)$ from Eq. (6.1) in the powers of β is $\mathcal{O}(1)$.*

Proof.

$$-[p_a(Q_{ab}+i\tilde{Q}_{ab})p_b - \frac{(p_a e_b(Q_{ab}+i\tilde{Q}_{ab})+i\Theta\cdot p_a n_a)^2}{\alpha_* + e_a e_b Q_{ab}}] \tag{7.70}$$

The imaginary part of the exponent after integration is given by:

$$-p_a\tilde{Q}_{ab}p_b + 2\frac{p_a e_b Q_{ab}(p_c e_f \tilde{Q}_{cf} + \Theta\cdot p_c n_c)}{\alpha_* + e_a e_b Q_{ab}} \tag{7.71}$$

In the above expresion we treat p and r_i in an unified fasion. The only restriction is that e_a is never associated to lines carrying momentum r_i. Furthermore, because the momenta r_i are conserved in pairs, they will not appear in the term $\Theta\cdot p_c n_c$. We now split r from p and write separately the $r_i.r_j$ and $r_i.p$:

$$-r_i\tilde{Q}_{ij}r_j + 2\frac{r_i e_b Q_{ib}\ r_j e_f \tilde{Q}_{jf}}{\alpha_* + e_a e_b Q_{ab}} \tag{7.72}$$

$$-2r_i\tilde{Q}_{ib}p_b + 2\frac{r_i e_b Q_{ib}(p_c e_f \tilde{Q}_{cf} + \Theta\cdot p_c n_c)}{\alpha_* + e_a e_b Q_{ab}} + 2\frac{p_a e_b Q_{ab}\ r_i e_f \tilde{Q}_{if}}{\alpha_* + e_a e_b Q_{ab}} \tag{7.73}$$

We will use induction to show that both the $r_i r_j$ and $r_i p_a$ terms are of order 1 on the β expansion.

By explicit computation it can be shown that all the 1-loop graphs satisfy this assumption. We proceed therefore by assuming that for all L-loop graphs the imaginary part of the $r_i r_j$ and $r_i p_a$ terms in the exponent are of order 1 and apply the two Bogoliubov operations on these graphs[27]. We will show that the resulting exponents satisfy the assumption.

It is easy to see that by performing the first operation, adding a vertex together with a line joining it to some graph G, we get the same scaling for the imaginary part as for G itself. (actually this case is covered by the assumption since the operation does not change the number of loops).

Before we proceed to the second operation, let us notice that it is enough to show that $(r_i e_b Q_{ib})/(\alpha_* + e_a e_b Q_{ab})$ and $(p_a e_b Q_{ib})/(\alpha_* + e_a e_b Q_{ab})$ are of order 1. It is actually easy to show that. Consider the first expression:

$(r_i e_b Q_{ib})/(\alpha_* + e_a e_b Q_{ab})$

We first notice that the leading term in the denominator comes from $e_a e_b Q_{ab}$. The reason is the following: look at the term with β_{I+1}. If the line α_* is in $K_2(G_{ab})$, then $e_a e_b Q_{ab}$ is proportional to $1/\beta_{I+1}^2$ thereas $\alpha_* \sim \beta_{I+1}^2$. If the line α_* is not in $K_2(G_{ab})$, then $e_a e_b Q_{ab}$ is proportional to β_{I+1}^0 thereas $\alpha_* \sim \beta_{I+1}^2$. So we need to study only

$$\frac{r_i e_b Q_{ib}}{e_a e_b Q_{ab}} \tag{7.74}$$

[27] The two Bogoliubov operations are: (1) a new vertex is added together with one line joining this vertex to one of the old vertices; (2) a new internal line is added for a given number of vertices.

Now, $e_a e_b Q_{ab}$ can be interpreted as the p^2 part of the exponent of the graph obtained from the original one by setting to zero all external momenta except those of the 2 lines e_a. From theorem 4 follows that $r_i e_b Q_{ib}$ is never leading over $e_a e_b Q_{ab}$.[28] The equations expressing this are a direct application of Eq. (7.48):

$$e_a e_b Q_{ab} = \left(\prod_{i=k}^{I+1} \frac{1}{\beta_i^2} \right) [1 + \mathcal{O}(\beta)] \tag{7.75}$$

$$r_i e_b Q_{ib} = f(\beta_1, ..., \beta_{k'-1}) \left(\prod_{i=k'}^{I+1} \frac{1}{\beta_i^2} \right) [1 + \mathcal{O}(\beta)] \tag{7.76}$$

with $k' \geq k$ and f being some polynomial. Thus, the ratio Eq. (7.74) is

$$f(\beta_1, ..., \beta_{k'-1}) \left(\prod_{i=k}^{k'-1} \beta_i^2 \right) [1 + \mathcal{O}(\beta)] \tag{7.77}$$

which in all situations is of order larger or equal to 1.

Combining this with the inductive assumption we see that this implies that the expresion in Eq. (7.72) is of order $\mathcal{O}(1)$ or higher.

Consider now the term $(p_a e_b Q_{ab})/(\alpha_* + e_a e_b Q_{ab})$. By the same argument as before it is enough to study the ratio

$$\frac{p_a e_b Q_{ab}}{e_a e_b Q_{ab}} \tag{7.78}$$

Using theorem 4 and the remark following it on the numerator of Eq. (7.78) we see that each term is, after summation over e_b, not leading over $\sum_{a,b} e_a Q_{ab} e_b$. In formulas this means:

$$e_a e_b Q_{ab} = \left(\prod_{i=k}^{I+1} \frac{1}{\beta_i^2} \right) [1 + \mathcal{O}(\beta)] \tag{7.79}$$

$$p_a e_b Q_{ab} = g(\beta_1, ..., \beta_{k'-1}, p_a) \left(\prod_{i=k'}^{I+1} \frac{1}{\beta_i^2} \right) [1 + \mathcal{O}(\beta)] \tag{7.80}$$

with $k' \geq k$ and g some polynomial. Thus, the ratio Eq. (7.78) is

$$g(\beta_1, ..., \beta_{k'-1}, p_a) \left(\prod_{i=k}^{k'-1} \beta_i^2 \right) [1 + \mathcal{O}(\beta)] \tag{7.81}$$

which is of order larger or equal to 1.

Combining this with the inductive assumption we see that this implies that the expresion in Eq. (7.73) is of order larger or equal to 1.

We have therefore shown that all the $L + 1$ loop graphs have the part proportional to r_i in the imaginary part of the exponent of order β^0 and higher in the β expansion. **q.e.d.**

[28] This is just the application of theorem 4 to the particular choice of external momenta $p_a = pe_a$.

Lemma 7.9 *Let $\gamma_{k_{ij}}$ be the smallest subgraph in Eq. (5.4) which contains both of the lines $i \notin K_2$ and $j \notin K_2$ in the loops (if $i = j$ then $\gamma_{k_{ij}}$ is the smallest subgraph in Eq. (5.4) which contains i in a loop). Then, the leading r^2 term in the exponent of Eq. (7.5) in the β expansion is of the form*

$$\left(r_i(\Theta^2) r_j \prod_{l=k_2}^{I} \beta_l^{-2} \right) R_2(\beta_1, \beta_2, \ldots, \beta_{k_2-1}) \tag{7.82}$$

for some $k_2 \geq k_{ij}$, where R_2 is a polynomial in variables $\beta_1, \beta_2, \ldots, \beta_{k_2-1}$.

Proof.

Consider first the case $i = j$. Let $\imath', \imath'' \in \mathcal{E}'(G)$ be the external lines attached to the internal line $i \notin K_2$. Applying theorem 1 to the numerator and the denominator of the ratio $P(G_{\imath'\imath''}|\alpha_{\imath'\imath''} = 0)/P(G)$ as in theorem 2, we find

$$\frac{P(G_{\imath'\imath''}|\alpha_{\imath'\imath''} = 0)}{P(G)} = \theta^2 \, \frac{\prod_{l=1}^{I} \beta_l^{2(L(\gamma_l) - c_{G_{\imath'\imath''}}(\gamma_l))}}{\prod_{l=1}^{I} \beta_l^{2(L(\gamma_l) - c_G(\gamma_l))}} [1 + \mathcal{O}(\beta^2)] \tag{7.83}$$

It is easy to see that none of the subgraphs of graph G which do not contain line i in a loop will wrap the new cycle of graph $G_{\imath'\imath''}$ formed by the joining of lines $\imath', \imath''$. That is we have

$$c_{G_{\imath'\imath''}}(\gamma) = c_G(\gamma) \tag{7.84}$$

for such subgraphs. Thus, for some $k_2 \geq k_{ij}$, we have

$$c_{G_{\imath'\imath''}}(\gamma_l) = c_G(\gamma_l) + 1 \tag{7.85}$$

for all $l \geq k_2$ and

$$c_{G_{\imath'\imath''}}(\gamma_l) = c_G(\gamma_l) \tag{7.86}$$

for $1 \leq l \leq k_2$.[29] We thus conclude that Eq. (7.82) holds in the case $i = j$.

Consider now the case $i \neq j$. Let $\imath', \imath'' \in \mathcal{E}'(G)$ and $\jmath', \jmath'' \in \mathcal{E}'(G)$ be the external lines attached to the internal lines $i \notin K_2$ and $j \notin K_2$, respectively. Up to an overall factor, the contribution to the exponent in Eq. (7.5) reads

$$r_i r_j e_a Q_{ab} e_b = r_i r_j [P(G_{\imath'\jmath'}) - P(G_{\imath'\jmath''}) - P(G_{\imath''\jmath'}) + P(G_{\imath''\jmath''})]/P(G) \tag{7.87}$$

where $e_{\imath'} = 1$, $e_{\imath''} = -1$ and the rest are zero. Using theorem 4 and the remark following it we get that $Q_{\imath'\jmath'} - Q_{\imath'\jmath''}$ is not leading over $Q_{\jmath'\jmath''}$ and similarly for the other three differences of Q-s appearing in Eq. (7.87). Let γ_{s_i} and γ_{s_j} be the smallest subgraphs of G

[29] k_2 need not be equal to k_{ij} as the following example shows. Consider the graph in figure 45 and suppose that $i = 1$. Let us choose the identity permutation, i.e. $\alpha_1 = \beta_1^2 \beta_2^2 \beta_3^2$, $\alpha_2 = \beta_2^2 \beta_3^2$, $\alpha_3 = \beta_3^2$. The r.h.s. of Eq. (7.83) for this case reads: $\frac{\beta_1^{2(1-1)} \beta_2^{2(2-2)} \beta_3^{2(3-3)}}{\beta_1^{2(1-1)} \beta_2^{2(2-2)} \beta_3^{2(3-2)}} = \frac{1}{\beta_3^2}$. In this case we have $k_{ij} = 1$ and $k_2 = 3$.

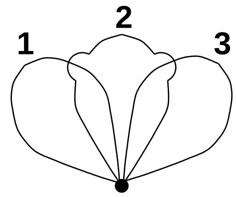


Figure 45: Graph for footnote 29

that contain either i or j in loops. There are three situations that can appear in a generic permutation:

- $\gamma_{s_i} \subset \gamma_{s_j}$ in which case $\gamma_{s_j} = \gamma_{k_{ij}}$ in the text of the lemma
- $\gamma_{s_j} \subset \gamma_{s_i}$ in which case $\gamma_{s_i} = \gamma_{k_{ij}}$ in the text of the lemma
- $\gamma_{s_i} = \gamma_{s_j}$

The first two situations are similar. For the first one, from theorem 4 we have that neither of $Q_{\imath'\jmath'} - Q_{\imath'\jmath''}$ nor $Q_{\imath''\jmath'} - Q_{\imath''\jmath''}$ is leading over $Q_{\jmath'\jmath''}$. Thus, we get that the leading term in the expansion for small β of Eq. (7.87) is

$$\left(r_i(\Theta^2) r_j \prod_{l=k_2}^{I} \beta_l^{-2} \right) R_2(\beta_1, \beta_2, \ldots, \beta_{k_2-1}) \tag{7.88}$$

with $k_2 \geq s_j = k_{ij}$.

For the second situation, from theorem 4 we have that neither of $Q_{\imath'\jmath'} - Q_{\imath''\jmath'}$ nor $Q_{\imath'\jmath''} - Q_{\imath''\jmath''}$ is leading over $Q_{\imath'\imath''}$. Thus, we get that the leading term in the expansion for small β of Eq. (7.87) is

$$\left(r_i(\Theta^2) r_j \prod_{l=k_2}^{I} \beta_l^{-2} \right) R_2(\beta_1, \beta_2, \ldots, \beta_{k_2-1}) \tag{7.89}$$

with $k_2 \geq s_i = k_{ij}$.

The third situation is much simpler since it does not require the use of the details of theorem 4. From the assumption that $\gamma_{s_i} = \gamma_{s_j}$ and theorem 2 we see that the leading term in the expansion of each of the four terms in Eq. (7.87) is given by

$$\left(r_i(\Theta^2) r_j \prod_{l=k_2}^{I} \beta_l^{-2} \right) \tag{7.90}$$

with $k_2 \geq s_i = s_j = k_{ij}$.

Combining the three cases, we conclude that Eq. (7.82) holds in the case $i \neq j$. **q.e.d.**

Theorem 5 (Convergence theorem) *Let I_G be the Feynman integral for a 1PI graph G in a field theory over non-commutative $\mathbb{R}^d$ with the massive propagators. There are three cases:*

(I) If $j(G) = 0$, then I_G is convergent if $\omega(\gamma) - c(\gamma)d < 0$ for all $\gamma \subseteq G$.

(II) If $j(G) = 1$ and the external momenta are non-exceptional, then I_G is convergent if for any subgraph $\gamma \subseteq G$ at least one of the following conditions is satisfied:(1) $\omega(\gamma) - c(\gamma)d < 0$, (2) $j(\gamma) = 1$.

(III) If $j(G) = 1$ and the external momenta are exceptional, then I_G is convergent if $\omega(\gamma) - c(\gamma)d < 0$ for all $\gamma \subseteq G$.

Proof.

This theorem follows from theorem 4 and lemmas 7.1, 7.8 and 7.9 as explained in section 7.1. **q.e.d.**

8 Comments

• For the massive theories the integral Eq. (6.1) is convergent at the upper limit ($\alpha \to \infty$) of the integration. In the massless case, the IR power counting theorem of ref.[17] does not hold in the non-commutative case. Consider the graph in figure 11. In the non-derivative massless scalar NQFT in $d = 4$ the corresponding Feynman integral reads

$$\int_0^\infty d\alpha_1 d\alpha_2 \frac{1}{(\alpha_1\alpha_2 + \theta^2)^2} = \int_0^\infty \frac{d\alpha_1}{\alpha_1} \int_0^\infty d\alpha_2 \frac{1}{(\alpha_2 + \theta^2)^2} \tag{8.1}$$

This integral is divergent. In the massless commutative theory the graph 11 is IR convergent and UV divergent. In the massless non-commutative theory it is both IR and UV divergent.

• The divergences from **Com** graphs can be removed by the mass renormalization as discussed in section 4. In the massless theories this procedure does not work.

• The techniques developed in this paper can be extended to study the asymptotic dependence of Feynman diagrams upon external momenta. In the commutative case this problem was studied in ref.[18].

• We formulated the convergence theorem for the nondegenerate matrix Θ. It is not difficult to see that in the degenerate case one has to modify the condition $\omega - cd < 0$ and the definition of exceptional external momenta. The former condition is modified to $\omega - c\ \mathrm{rank}\Theta < 0$. In the degenerate case, one has to restrict to the momenta along the nondegenerate directions in Eq. (3.4).

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Appendix A

Let us derive relation Eq. (6.41). Consider

$$Q = \sum_{i \neq j} p_i Q_{ij} p_j$$

and

$$\tilde{Q} = \sum_{i \neq j} p_i \tilde{Q}_{ij} p_j$$

from the exponent of Eq. (6.1). We can always write Q and $\tilde{Q}$ in this way by using momentum conservation.

Consider two arbitrary external lines i, j of a graph G and form a new graph G_{ij} as in figure 10. In terms of Feynman integral Eq. (6.1) for the graph G, it corresponds to setting $p_i = -p_j = q$, including a Schwinger parameter α_{ij} corresponding to the new line formed from the joining of lines i and j, and integrating over the momentum q. Since $\tilde{Q}_{ij}$ is anti-symmetric in the indices i and j, it does not contribute to the determinant coming from the integration over q. As a result, we find

$$P(G_{ij}) = P(G)(\alpha_{ij} + Q_{ij}) \tag{A.1}$$

Thus

$$Q_{ij} = \frac{P(G_{ij})|_{\alpha_{ij}=0}}{P(G)}$$

The proof of the inequality Eq. (6.42) is analogous to the one for the commutative theories given in ref.[19]. Relation Eq. (6.42) may be verified by the method of induction, since its validity in the case of simplest diagrams is evident. We shall carry out the induction by using the two Bogoliubov operations introduced in footnote 27: (1) a new vertex is added together with one line joining this vertex to one of the old vertices; (2) a new internal line is added for a given number of vertices.

It is easy to see that the inequality Eq. (6.42) holds after the operation (1). Thus consider operation (2). For the sake of definiteness we shall assume that the internal line with momentum k is added between vertices 1 and 2. The modified exponent reads

$$-\alpha_*\ k^2 - \sum_{a,b,a \neq b} (p_a + e_a k)(Q_{ab} + i\tilde{Q}_{ab})(p_b + e_b k) - i\Theta_{\mu\nu} n^a p_a^\mu k^\nu \tag{A.2}$$

where the symbol e_a is defined as follows

$$e_1 = 1, \quad e_2 = -1, \quad e_a = 0 \ \ (a > 2) \tag{A.3}$$

and n_a takes values ± 1 or 0 depending upon whether the line with momentum p_a is intersected or not by the added line $*$. As usual, the orientation of the intersection is given by the sign of n_a. Integrating over k, one finds

$$-[p_a(Q_{ab} + i\tilde{Q}_{ab})p_b - \frac{(p_a e_b(Q_{ab} + i\tilde{Q}_{ab}) + i\Theta \cdot p_a n_a)^2}{\alpha_* + e_a e_b Q_{ab}}] \tag{A.4}$$

Now we need to show that the real part of minus this expression is larger or equal to zero, i.e.

$$p_a Q_{ab} p_b - \frac{(p_a e_b Q_{ab})^2}{\alpha_* + e_a e_b Q_{ab}} + \frac{(\Theta \cdot p_a n_a + p_a e_b \tilde{Q}_{ab})^2}{\alpha_* + e_a e_b Q_{ab}} \geq 0 \tag{A.5}$$

The last term in the above expression is positive since it is the norm of a real vector. By the inductive assumption we have

$$Q_{ab} q_a q_b \geq 0$$

for any q_a. As in the commutative case ref.[19], it implies that

$$(p_a Q_{ab} p_b)(e_a Q_{ab} e_b) - (e_a Q_{ab} p_b)^2 \geq 0 \tag{A.6}$$

Combining Eq. (A.5) and Eq. (A.6) we arrive at the conclusion that the real part of the exponent obtained by adding a line to a graph is non-negative.

References

[1] T. Eguchi and R. Nakayama, *Simplification of Quenching Procedure For Large N Spin Models*, Phys.Lett. B122 (1983) 59;

A. Gonzalez-Arroyo and M. Okawa, *Twisted Eguchi-Kawai model: A reduced model for large-N lattice gauge theory*, Phys.Rev.D27:2397,1983;

A. Gonzalez-Arroyo and C.P. Korthals Altes, *Reduced model for large N continuum theories*, Phys.Lett.B131:396,1983.

[2] T. Filk, Phys.Lett. B376 (1996) 53-58.

[3] J.C. Varilly and J.M. Gracia-Bondia, Int.J.Mod.Phys. A14 (1999) 1305, hep-th/9804001;

M. Chaichian, A. Demichev and P. Presnajder, hep-th/9812180, hep-th/9904132;

C.P. Martin and D. Sanchez-Ruiz, Phys.Rev.Lett. 83 (1999) 476-479, hep-th/9903077;

M.M. Sheikh-Jabbari, JHEP 9906 (1999) 015, hep-th/9903107;

T. Krajewski and R. Wulkenhaar, hep-th/9903187;

S. Cho, R. Hinterding, J. Madore and H. Steinacker, hep-th/9903239;

D. Bigatti and L. Susskind, hep-th/9908056;

H.B. Benaoum, hep-th/9912036;

C.S. Chu and F. Zamora, JHEP 0002 (2000) 022, hep-th/9912153;

S. Iso, H. Kawai and Y. Kitazawa, hep-th/0001027;

H. Grosse, T. Krajewski and R. Wulkenhaar, hep-th/0001182;

W. Fischler, J. Gomis, E. Gorbatov, A. Kashani-Poor, S. Paban and P. Pouliot, JHEP 0005 (2000) 024, hep-th/0002067;

J.M. Gracia-Bondia and C.P. Martin, Phys.Lett. B479 (2000) 321-328, hep-th/0002171;

M.V. Raamsdonk and N. Seiberg, JHEP 0003 (2000) 035, hep-th/0002186.

R. Oeckl, Nucl.Phys. B581 (2000) 559-574, hep-th/0003018;

W. Fischler, E. Gorbatov, A. Kashani-Poor, R. McNees, S. Paban and P. Pouliot, JHEP 0006 (2000) 032, hep-th/0003216;

B.A. Campbell and K. Kaminsky, Nucl.Phys. B581 (2000) 240-256, hep-th/0003137;

F. Zamora, JHEP 0005 (2000) 002, hep-th/0004085;

J. Gomis, K. Landsteiner and E.Lopez, hep-th/0004115;

I. Mocioiu, M. Pospelov and R. Roiban, hep-ph/0005191;

A. Armoni, hep-th/0005208;

S. Kar, hep-th/0006073, hep-th/9911251;

S.S. Gubser and S.L. Sondhi, hep-th/0006119;

C.P. Martin and F. Ruiz Ruiz, hep-th/0007131;

[4] I. Chepelev and R. Roiban, JHEP 0005 (2000) 037, hep-th/9911098.

[5] S. Minwalla, M.V. Raamsdonk and N. Seiberg, JHEP 0002 (2000) 020, hep-th/9912072.

[6] I.Ya. Aref'eva, D.M. Belov and A.S. Koshelev, Phys.Lett. B476 (2000) 431-436, hep-th/9912075.

[7] I.Ya. Aref'eva, D.M. Belov and A.S. Koshelev, hep-th/0001215;

[8] C.S. Chu and F. Zamora, JHEP 0002 (2000) 022, hep-th/9912153; S. Ferrara and M.A. Lledo, JHEP 0005 (2000) 008, hep-th/0002084; S. Terashima, Phys.Lett. B482 (2000) 276-282, hep-th/0002119; A.A. Bichl, J.M. Grimstrup, H. Grosse, L. Popp, M. Schweda and R. Wulkenhaar, hep-th/0007050.

[9] A. Matusis, L. Susskind and N. Toumbas, hep-th/0002075.

[10] I.Ya. Aref'eva, D.M. Belov, A.S. Koshelev and O.A. Rytchkov, hep-th/0003176.

[11] O. Andreev and H. Dorn, hep-th/0003113; Y. Kiem and S. Lee, hep-th/0003145; A. Bilal, C.S. Chu and R. Russo, hep-th/0003180; J. Gomis, M. Kleban, T. Mehen, M. Rangamani and S. Shenker, hep-th/0003215. C.-S. Chu, R. Russo and S. Sciuto, hep-th/0004183; Y. Kiem, S. Lee and J. Park, hep-th/0008002;

[12] H.O. Girotti, M. Gomes, V.O. Rivelles and A.J. da Silva, hep-th/0005272.

[13] An (incomplete) list of references is:

A. Connes, M.R. Douglas and A. Schwarz, JHEP 9802 (1998) 003, hep-th/9711162;

N. Nekrasov and A. Schwarz, Commun.Math.Phys. 198 (1998) 689-703, hep-th/9802068;

H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Nucl.Phys. B565 (2000) 176-192, hep-th/9908141;

M. Li, Nucl.Phys. B499 (1997) 149-158, hep-th/9612222;

N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl.Phys. B573 (2000) 573-593, hep-th/9910004;

I. Bars and D. Minic, hep-th/9910091;

J. Ambjorn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, JHEP 9911 (1999) 029, hep-th/9911041;

S. Iso, H. Kawai and Y. Kitazawa, Nucl.Phys. B576 (2000) 375-398, hep-th/0001027;

R. Gopakumar, S. Minwalla and A. Strominger, JHEP 0005 (2000) 020, hep-th/0003160;

N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, hep-th/0004038;

J. Ambjorn, Y.M. Makeenko, J. Nishimura and R.J. Szabo, JHEP 0005 (2000) 023, hep-th/0004147;

L. Alvarez-Gaume and J.L.F. Barbon, hep-th/0006209;

A.S. Gorsky, Y.M. Makeenko and K.G. Selivanov, hep-th/0007247;

C.-G. Zhou, hep-th/0007255;

[14] G. Springer, *Introduction to Riemann Surfaces*, New York 1981.

[15] C. Itzykson and J.-B. Zuber,(1980) *Quantum Field Theory* (McGraw-Hill, New York).

[16] N. Nakanishi, *Graph Theory and Feynman Integrals* (Gordon and Breach).

[17] J.H. Lowenstein and W. Zimmermann, Commun. Math. Phys. 44, 73 (1975).

[18] S. Weinberg, Phys.Rev. 118, 838 (1960).

[19] N.N. Bogoliubov and D.V. Shirkov,(1959) *Introduction to The Theory of Quantized Fields.*